On locally AH algebras

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Abstract

We show that every unital amenable separable simple C^*-algebra with finite tracial rank which satisfies the UCT has in fact tracial rank at most one. We also show that unital separable simple C^*-algebras which are "tracially" locally AH with slow dimension growth are $\mathcal{Z}$-stable. As a consequence, unital separable simple C^*-algebras which are locally AH with no dimension growth are isomorphic to a unital simple AH-algebra with no dimension growth.

1 Introduction

The program of classification of amenable C^*-algebras, or the Elliott program, is to classify amenable C^*-algebras up to isomorphisms by their K-theoretical data. One of the high lights of the success of the Elliott program is the classification of unital simple AH-algebras (inductive limits of homogeneous C^*-algebras) with no dimension growth by their K-theoretical data (known as the Elliott invariant) ([16]). The proof of this first appeared near the end of the last century. Immediately after the proof appeared, among many questions raised is the question whether the same result holds for unital simple locally AH-algebras (see the definition 3.5 below) with no dimension growth. It should be noted that AF-algebras is locally finite dimensional. But (separable) AF-algebras are inductive limits of finite dimensional C^*-algebras. The so-called $A\mathbb{T}$-algebras are inductive limits of circle algebras. More than often, these $A\mathbb{T}$-algebras arise as local circle algebras (approximated by circle algebras). Fortunately, due to the weak-semi-projectivity of circle algebras, locally $A\mathbb{T}$-algebras are $A\mathbb{T}$-algebras. However, the situation is completely different for locally AH algebras. In fact it was proved in [11] that there are unital C^*-algebras which are inductive limits of AH-algebras but themselves are not AH-algebras. So in general, a locally AH algebra is not an AH algebra.

On the other hand, however, it was proved in [26] that a unital separable simple C^*-algebra which is locally AH is a unital simple AH-algebra, if, in addition, it has real rank zero, stable rank one and weakly unperforated K_0-group and which has countably many extremal traces. In fact these C^*-algebras have tracial rank zero. The tracial condition was late removed in [58]. In particular, if A is a unital separable simple C^*-algebra which is locally AH with no (or slow) dimension growth and which has real rank zero must be a unital AH-algebra. In fact such C^*-algebras have stable rank one and have weakly unperforated $K_0(A)$. The condition of real rank zero forces these C^*-algebras to have tracial rank zero. More recently, classification theory extends to those C^*-algebras that are rationally tracial rank at most one ([59], [32], [37] and [35]). These are unital separable simple amenable C^*-algebras A such that $A\otimes U$ have tracial rank at most one for some infinite dimensional UHF algebra U. An important subclass of this (which includes, for example, the Jiang-Su algebra $\mathcal{Z}$) is the class of those unital separable simple C^*-algebras A such that $A\otimes U$ have tracial rank zero. By now we have some mechanical tools to verify certain C^*-algebras to have tracial rank zero (see [26], [5], [58] and [38]) and based on these results, we have some tools to verify when a unital simple C^*-algebra is rationally tracial rank zero ([55] and [54]). However, these result could not be applied to the case that C^*-algebras are of tracial rank one, or rationally tracial rank one. Until now, there is no effective way, besides Gong's decomposition result ([18]), to verify when a unital separable simple C^*-algebra has tracial rank one (but not tracial rank zero). In fact, as mentioned above, we did not even know when a unital simple separable locally AH algebra with no dimension growth has tracial rank one. This makes it much hard to decide when a unital simple separable C^*-algebra is rationally tracial rank one.

Closely related problem is whether a unital separable simple C^*-algebra with finite tracial rank is in fact of tracial rank at most one. This is an open problem for a decade. If the problem has an affirmative answer, it will make easier, for many cases, to decide certain unital simple C^*-algebras to have tracial rank at most one.

The purpose of this research is to solve these problems. Our main results include the following:

Theorem 1.1. *Let A be a unital separable simple C^*-algebra which is locally AH with no dimension growth. Then A is isomorphic to a unital simple AH-algebra with no dimension growth.*

We actually prove the following.

Theorem 1.2. *Let A be a unital separable simple amenable C^*-algebra with finite tracial rank which satisfies the Universal Coefficient Theorem. Then A is isomorphic to a unital simple AH-algebra with no dimension growth. In particular, A has tracial rank at most one.*

To establish the above, we also prove the following

Theorem 1.3. *Let A be a unital separable simple C^*-algebra in $\mathcal{C}_1$ then A is $\mathcal{Z}$-stable, i.e., $A \cong A \otimes \mathcal{Z}$.*
(See 3.6 below for the definition of $\mathcal{C}_1$.)

The article is organized as follows. Section 2 serves as a preliminary which includes a number of conventions that will be used throughout this article. Some facts about a subgroup $SU(M_n(C(X))/CU(M_n(C(X)))$ have been discussed. The detection of those unitaries with trivial determinant at each point which are not in the closure of commutator subgroup play new role in the Basic Homotopy Lemma which will be presneted in section 11. In section 3, we introduce the class $\mathcal{C}_1$ of simple C^*-algebras which may be described as tracially locally AH algebras of slow dimension growth. Several related definitions are given. In section 4, we discuss some basic properties of C^*-algebras in class $\mathcal{C}_1$. In section 5, we prove, among other things, that C^*-algebras in $\mathcal{C}_1$ have stable rank one and the strict comparison for positive elements. In section 6, we study the tracial state space of a unital simple C^*-algebra in $\mathcal{C}_1$. In particular, we show that every quasi-trace of a unital separable simple C^*-algebra in $\mathcal{C}_1$ extends to a trace. Moreover, we show that, for a unital simple C^*-algebra A in $\mathcal{C}_1$, the affine map from the tracial state space to state space of $K_0(A)$ maps the extremal points onto the extremal points. In section 7, we discuss the unitary groups of simple C^*-algebras in a subclass of $\mathcal{C}_1$. In section 8, using what have been established in previous sections, we combine an argument of Winter ([60])and an argument of Matui and Sato ([39]) to prove Theorem 1.2 above. In section 9 we present some versions of so-called existence theorem. In section 10, we present a uniqueness statement that will be proved in section 12 and an existence type result regarding the Bott map. The uniqueness theorem holds for Y being a finite CW complex of dimension zero as well as the case that $Y = [0,1]$. An induction on the dimension d will be presented in the next two sections. In section 11, we present a version of The Basic Homtopy Lemma which was first studied intensively in [4] and later in [31]. A new obstruction for the Basic Homotopy Lemma in this version will be dealt with which was earlier mentioned in section 2. In section 12, we prove the uniqueness statement in section 10. In section 13 we present the proofs for Theorem 1.1 and 1.3. Section 14 serves an appendix to this article.

2 Preliminaries

2.1. Let A be a unital C^*-algebra. Denote by $T(A)$ the convex set of tracial states of C. Let $\mathrm{Aff}(T(A))$ be the space of all real affine continuous functions on $T(A)$. Denote by $M_n(A)$ the algebra of all $n\times n$ matrixes over A. By regarding $M_n(A)$ as a subset of $M_{n+1}(A)$, define $M_\infty(A)=\cup_{n=1}^\infty M_n(A)$. If $\tau\in T(A)$, then $\tau\otimes \mathrm{Tr}$, where Tr is standard trace on M_n, is a trace on $M_n(A)$. Throughout this paper, we will use τ for $\tau\otimes\mathrm{Tr}$ without warning.

We also use $QT(A)$ for the set of all quasi-traces of A.

Let C and A be two unital C^*-algebras with $T(C)\neq\emptyset$ and $T(A)\neq\emptyset$. Suppose that $h: C\to A$ is a unital homomorphism. Define an affine continuous map $h_\sharp: T(A)\to T(C)$ by $h_\sharp(\tau)(c)=\tau\circ h(c)$ for all $\tau\in T(A)$ and $c\in C$.

Definition 2.2. Let C be a unital C^*-algebra with $T(C)\neq\emptyset$. For each $p\in M_n(C)$ define $\check{p}(\tau)=(\tau\otimes\mathrm{Tr})(p)$ for all $\tau\in T(A)$, where Tr is the standard trace on M_n. This gives positive homomorphism $\rho_C: K_0(C)\to\mathrm{Aff}(T(C))$.

A positive homomorphism $s: K_0(A)\to\mathbb{C}$ is a state on $K_0(A)$ if $s([1_A])=1$. Let $S(K_0(A))$ be the state space of $K_0(A)$. Define $r_A: T(C)\to S(K_0(A))$ by $r_A(\tau)([p])=\tau(p)$ for all projections $p\in M_n(A)$ (for all $n\geq 1$).

Definition 2.3. Let A and B be two C^*-algebras and $\varphi: A\to B$ be a positive linear map. We will use $\varphi^{(K)}: A\to M_K(B)$ for the map $\Phi^{(K)}(a)=\varphi(a)\otimes 1_{M_K}$. If $a\in B$, we may write $a^{(K)}$ for the element $a\otimes 1_{M_K}$ and sometime it will be written as $\mathrm{diag}(\overbrace{a,a,...,a}^{K})$.

Definition 2.4. In what follows, we may identify $\mathbb{T}$ with the unit circle and $z\in C(\mathbb{T})$ with the identity map on the circle.

Definition 2.5. Let A be a unital C^*-algebra. Following [17], define

$$F_nK_i(A)=\{\varphi_{*i}(z_b)\in K_i(A):\varphi\in Hom(C(S^n),M_\infty(A))\},$$

where z_b is a generator (the bott element) of $K_i(C(S^n))$, if $i=0$ and n even, or $i=1$ and n odd. $F_nK_i(A)$ is a subgroup of $K_i(A)$, $i=0,1$.

Definition 2.6. Fix an integer $n\geq 2$, let z be a generator of $K_1(C(S^{2n-1}))$. Let z_b be a unitary in $M_n(C(S^{2n-1}))$ which represents z. We fix one such unitary that $z_b\in SU_n(C(S^{2n-1}))$, i.e., $\det(u(x))=1$ for all $x\in S^{2n-1}$. In case $n=2$, one may write

$$z_b=\begin{pmatrix} z & -\bar{w}\\ w & \bar{z}\end{pmatrix}, \tag{e 2.1}$$

where $S^3=\{(z,w)\in\mathbb{C}^2: |z|^2+|w|^2=1\}$.

2.7. Let C be a unital C^*-algebra. Denote by $U(C)$ the unitary group of C and denote by $U_0(C)$ the subgroup of $U(C)$ consisting of unitaries which connected to 1_C by a continuous path of unitaries. Denote by $CU(C)$ the closure of the normal subgroup generated by commutators of $U(C)$. Let $u\in U(C)$. Then $\bar{u}$ is the image of u in $U(C)/CU(C)$. Let $\mathcal{W}\subset U(A)$ be a subset. Denote by $\overline{\mathcal{W}}$ the set of those elements $\bar{u}$ such that $u\in\mathcal{W}$. Denote by $CU_0(C)$ the intersection $CU(C)\cap U_0(C)$. Note that $U(A)/CU(A)$ is an abelian group.

We use the following metric on $U(A)/CU(A)$:

$$\mathrm{dist}(\bar{u},\bar{v})=\inf\|uv^*-w\|: w\in CU(A)\}.$$

Using de la Harp-Scandalis determinant, by K. Thomsen (see [52]), there is a short splitting exact sequence

$$0 \to \mathrm{Aff}(T(C))/\overline{\rho_C(K_0(C))} \to \cup_{n=1}^{\infty} U(M_n(C))/CU(M_n(C)) \to K_1(C) \to 0. \tag{e 2.2}$$

Suppose that $r \geq 1$ is an integer and $U(M_r(A))/U(M_r(A))_0 = K_1(A)$, one has the following short splitting exact sequence:

$$0 \to \mathrm{Aff}(T(C))/\overline{\rho_C(K_0(C))} \to U(M_r(C))/CU(M_r(C)) \to K_1(C) \to 0. \tag{e 2.3}$$

For $u \in U_0(C)$, we will use $\overline{\Delta}(u)$ for the de la Harp and Skandalis determinant of u, i.e., the image of u in $\mathrm{Aff}(T(C))/\overline{\rho_C(K_0(C))}$. For each C^*-algebra C with $U(C)/U_0(C) = K_1(C)$, we will fix one splitting map $J_c : K_1(C) \to U(C)/CU(C)$. For each $\bar{u} \in J_c(K_1(C))$, select and fix one element $u_c \in U(C)$ such that $\overline{u_c} = \bar{u}$. Denote this set by $U_c(K_1(C))$. Denote by $\Pi_c : U(C)/CU(C) \to K_1(C)$ the quotient map. Note that $\Pi_c \circ J_c = \mathrm{id}_{K_1(C)}$.

If A is a unital C^*-algebra and $\varphi : C \to A$ is a unital homomorphism, then φ induces a continuous homomorphism

$$\varphi^{\ddagger} : U(C)/CU(C) \to U(A)/CU(A).$$

If $g \in \mathrm{Aff}(T(A))$, denote by $\overline{g}$ the image of g in $\mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$.

Definition 2.8. Let A and B be two unital C^*-algebras. Let $G_1 \subset U(M_m(A))/CU(M_m(A))$ be a subgroup. Let $\gamma : G_1 \to U(M_m(B))/CU(M_m(B))$ be a homomorphism and let $\Gamma : \mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$ be an affine homomorphism. We say that Γ and λ are compatible if $\gamma(\bar{g}) = \overline{\Gamma(g)}$ for all $g \in \mathrm{Aff}(T(A))$ such that $\overline{g} \in G_1 \cap U_0(M_m(B))/CU(M_m(B)) \subset \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$. Let $\lambda : T(B) \to T(A)$ be continuous affine map. We say γ and λ are compatible if γ and the map from $\mathrm{Aff}(T(A)) \to \mathrm{Aff}(T(B))$ induced by λ are compatible. Let $\kappa \in Hom_{\Lambda}(\underline{K}(A), \underline{K}(B))$. We say that κ and γ are compatible if $\kappa|_{K_1(A)}(z) = \Pi_c \circ \gamma(z)$ for all $z \in G_1$. We say that κ and λ are compatible if $\rho_B(\kappa|_{K_0(A)}([p]) = \lambda(\tau)([p])$ for all projections $p \in M_{\infty}(A)$.

Definition 2.9. Let X be a compact metric space and let $P \in M_m(C(X))$ be a projection in $M_m(C(X))$ such that $P(x) \neq 0$ for all $x \in X$, where $m \geq 1$ is an integer. Let $C = PM_m(C(X))P$ and let $r \geq 1$ be an integer. Denote by $SU_r(C)$ the set of those unitaries $u \in M_r(C)$ such that $\det(u(x)) = 1$ for all $x \in X$. Note that $SU_r(C)$ is a normal subgroup of $U_r(C)$.

The following is an easy fact.

Proposition 2.10. *Let C be as in 2.9, let Y be a compact metric space and let $P_1 \in M_n(C(Y))$ be a projection such that $P_1(y) \neq 0$ for all $y \in Y$. Let $B = P_1 M_n(C(Y))P_1$. Suppose that $\varphi : C \to B$ is a unital homomorphism. Then φ maps $SU_r(C)$ into $SU_r(B)$ for all integer $r \geq 1$.*

It is also easy to see that $CU(M_r(C(C))) \subset \cup_{k=1}^{\infty} SU_k(C) \cap U_0(M_k(C))$. Moreover, one has the following:

Proposition 2.11. *Let X be a compact metric space and let $C = PM_m(C(X))P$ be as in 2.9. Then*

$$SU_r(C) \cap U_0(M_r(C)) \subset CU(M_r(C)) \ \text{for all} \ \ \textit{integer}\ r \geq 1.$$

Proof. Let $u \in SU_r(C) \cap U_0(M_r(C))$. Write $u = \prod_{j=1}^{k} \exp(\sqrt{-1}h_j)$, where $h_j \in M_r(C)_{s.a.}$. Put $R(x) = \mathrm{rank}\, P(x)$ for all $x \in X$. Note $R(x) \neq 0$ for all $x \in X$. It follows that

$$(\frac{1}{2\pi\sqrt{-1}})T_x(\sum_{j=1}^{k} h_j(x)) = N(x) \in \mathbb{Z}, \tag{e 2.4}$$

where T_x is the standard trace on $M_{rR(x)}$. Note that $N(x) \in C(X)$. Therefore there is a projection $Q \in M_L(C)$ such that

$$\mathrm{rank}Q(x) = N(x) \text{ for all } x \in X. \tag{e 2.5}$$

Let $\tau \in T(C)$. Then

$$\tau(f) = \int_X t_x(f) d\mu_\tau \text{ for all } f \in PM_m(C(X))P, \tag{e 2.6}$$

where t_x is the normalized trace on $M_{R(x)}$ and μ_τ is a Borel probability measure on X. Let Tr be the standard trace on M_L. Then

$$\rho_C(Q)(\tau) = \int_X (t_x \otimes Tr)(Q(x)) d\mu_\tau \tag{e 2.7}$$

$$= \int_X \frac{N(x)}{R(x)} d\mu_\tau \tag{e 2.8}$$

for all $\tau \in T(C)$, Define a smooth path of unitaries $u(t) = \prod_{j=1}^k \exp(\sqrt{-1}h_j(1-t))$ for $t \in [0,1]$. So $u(0) = u$ and $u(1) = \mathrm{id}_{M_r(C)}$. Then, with T being the standard trace on M_r,

$$(\frac{1}{2\pi\sqrt{-1}}) \int_0^1 (\tau \otimes T) \frac{du(t)}{dt} u(t)^* dt = (\frac{1}{2\pi\sqrt{-1}}) \int_0^1 (\tau \otimes T)(\sum_{j=1}^k h_j) dt \tag{e 2.9}$$

$$= (\frac{1}{2\pi\sqrt{-1}}) \int_X (t_x \otimes T)(\sum_{j=1}^k h_j(x)) d\mu_\tau = (\frac{1}{2\pi\sqrt{-1}}) \int_X \frac{T_x(\sum_{j=1}^k h_j(x))}{R(x)} d\mu_\tau \tag{e 2.10}$$

$$= \int_X \frac{N(x)}{R(x)} d\mu_\tau = \rho_C(Q)(\tau) \text{ for all } \tau \in T(C). \tag{e 2.11}$$

By a result of Thomsen ([52]), this implies that

$$\overline{u} = \overline{1} \in U(M_r(C)))/CU(M_r(C).$$

In other words,

$$SU_r(C) \cap U_0(C) \subset CU(M_r(C)).$$

□

Definition 2.12. Let X be a finite CW complex. Let $X^{(n)}$ be the n-skeleton of X and let $s_n : C(X) \to C(X^{(n)})$ be the surjective map induced by restriction, i.e., $s_n(f)(y) = f(y)$ for all $y \in X^{(n)}$. Let $P \in M_l(C(X))$ be a projection for some integer $l \geq 1$ and let $C = PM_l(C(Y))P$. Denote still by $s_n : C \to P^{(n)}M_l(C(X^{(n)}))P^{(n)}$, where $P^{(n)} = P|_{X^{(n)}}$. Put $C_n = P^{(n)}M_l(C(X^{(n)}))P^{(n)}$. Note that C_n is a quotient of C, and C_{n-1} is a quotient of C_n. It was proved by Exel and Loring ([17]) that $F_nK_i(C) = \ker(s_{n-1})_{*i}$.

Suppose that X has dimension N. Let

$$I_N = \ker r_{N-1} = \{f \in C : f|_{X^{(N-1)}} = 0\}.$$

Then I_N is an ideal of C. There is an embedding $j_N : \widetilde{I_N} \to C$ which maps $\lambda \cdot 1 + f$ to $\lambda \cdot 1_C + f$ for all $f \in I_N$. Define, for $1 < n < N$,

$$I_n = \{f \in C_n : f|_{X^{(n-1)}} = 0\}.$$

Again, there is an embedding $j_n : \widetilde{I_n} \to C_n$.

Note that $\widetilde{I_n} \cong P^{(n)'}M_l(C(Y_n))P^{(n)'}$, where $Y_n = S^n \bigvee S^n \bigvee \cdots \bigvee S^n$ (there are only finitely many of S^n).

Lemma 2.13. *Let X be a compact metric space. Then*

$$Tor(K_1(C(X))) \subset F_3K_1(C(X)). \tag{e 2.12}$$

Proof. We first consider the case that X is a finite CW complex. Let Y be the 2-skeleton of X and let $s : C(X) \to C(Y)$ be the surjective homomorphism defined by $f \mapsto f|_Y$ for $f \in C(X)$. Then, by Theorem 4.1 of [17],

$$F_3K_1(C(X)) = \mathrm{ker} s_{*1}. \tag{e 2.13}$$

Since $K_1(C(Y))$ is torsion free, $Tor(K_1(C(X))) \subset \mathrm{ker} s_{*1}$. Therefore

$$Tor(K_1(C(X))) \subset F_3K_1(C(X)). \tag{e 2.14}$$

For the general case, let $g \in Tor(K_1(C(X))$ be a non-zero element. Write $C(X) = \lim_{n\to\infty}(C(X_n), \varphi_n)$, where each X_n is a finite CW complex. There is n_0 and $g' \in K_1(C(X_{n_0})$ such that $(\varphi_{n_0,\infty})_{*1}(g') = g$. Let G_1 be the subgroup generated by g'. There is $n_1 \geq n_0$ such that $(\varphi_{n_1,\infty})_{*1}$ is injective on $(\varphi_{n_0,n_1})_{*1}(G_1)$. Let $g_1 = (\varphi_{n_0,n_1})_{*1}(g')$. Put $G_2 = (\varphi_{n_0,n_1})_{*1}(G_1)$. Then $G_2 \subset Tor(K_1(C(X_{n_1})))$. From what has been proved, $G_2 \in F_3K_1(C(X_{n_1}))$. It follows from part (c) of Proposition 5.1 of [17] that $\varphi_{n_1,\infty}(G_2) \subset F_3K_1(C(X))$. It follows that $g \in F_3K_1(C(X))$.

□

Lemma 2.14. *Let X be a compact metric space and let $G \subset K_1(C(X))$ be a finitely generated subgroup. Then $G = G_1 \oplus G \cap F_3K_1(C(X))$, where G_1 is a finitely generated free group.*

Proof. As in the proof of 2.13 we may assume that X is a finite CW complex. Let Y be the 2-skeleton of X. Let $s : C(X) \to C(Y)$ be the surjective map defined by the restriction $s(f)(y) = f(y)$ for all $f \in C(X)$ and $y \in Y$. Then, by Theorem 4.1 of [17], $\mathrm{ker} s_{*1} = F_3K_1(C(X))$. Therefore $G/G \cap F_3K_1(C(X))$ is isomorphic to a subgroup of $K_1(C(Y))$. Since $\dim Y = 2$, $Tor(K_1(C(Y))) = \{0\}$. Therefore $G/G \cap F_3K_1(C(X))$ is free. It follows that

$$G = G_1 \oplus G \cap F_3K_1(C(X))$$

for some finitely generated subgroup G_1. □

Definition 2.15. Let C be a unital C^*-algebra and let $G \subset K_1(C)$ be a finitely generated subgroup. Denote by $J' : K_1(C) \to \cup_{n=1}^{\infty} U(M_n(C))/CU(M_n(C))$ an injective homomorphism such that $\Pi \circ J' = \mathrm{id}_{K_1(C)}$, where Π is the surjective map from $\cup_{n=1}^{\infty} U(M_n(C))/CU(M_n(C))$ onto $K_1(C)$. There is an integer $N = N(G)$ such that $J'(G) \in U(M_N(C))/CU(M_N(C))$.

Let X be a compact metric space and let $C = PM_m(C(X))P$, where $P \in M_m(C(X))$ is a projection such that $P(x) \neq 0$ for all $x \in X$. By 2.14, one may write $G = G_1 \oplus G_b \oplus Tor(G)$, where G_b is the free part of $G \cap F_3K_1(C)$. Note, by 2.13, $Tor(G) \subset F_3K_1(C)$. Let $g \in Tor(G)$ be a non-zero element and let $\overline{u_g} = J'(g)$ for some unitary $u_g \in U(M_N(C))$. Suppose that $kg = 0$ for some integer $k > 1$. Therefore $u_g^k \in CU(M_N(C))$. It follows from 2.13 as well as 2.6, there are $h_1, h_2, ..., h_s \in M_{N+r}(C)_{s.a.}$ such that

$$u_1 \prod_{j=1}^{s} \exp(\sqrt{-1}h_j) \in SU_{r+N}(C),$$

where $u_1 = 1_{M_r} \oplus u$ and $r \geq 0$ is an integer. For each $x \in X$, $[\mathrm{det} u_1(x)]^k = 1$. It follows that

$$\frac{\sum_{j=1}^{s} kTr(h_j)(x)}{2\pi\sqrt{-1}} = I(x) \in \mathbb{Z} \ \text{ for all } \ x \in X.$$

It follows that $I(x) \in C(X)$. Therefore

$$(\prod_{j=1}^{s} \exp(\sqrt{-1}h_j))^k \in SU_{r+N}(C) \cap U_0(M_{r+N}(C)) \subset CU_{r+N}(C).$$

Consequently,

$$\overline{(u_1 \prod_{j=1}^{s} \exp(\sqrt{-1}h_j))^k} = \overline{1_{M_{r+N}(C)}}.$$

Thus there is an integer $R(G) \geq 1$ and an injective homomorphism

$$J_{c(G)} : Tor(G) \to SU_{R(G)}(C)/CU(M_{R(G)}(C))$$

such that $\Pi \circ J_{c(G)} = \mathrm{id}_{Tor(G)}$. By choosing a larger $R(G)$, if necessarily, one obtains an injective homomorphism $J_{c(G)} : G \to U(M_{R(G)}(C))/CU(M_{R(G)}(C))$ such that

$$J_{c(G)}(G_b \oplus Tor(G)) \subset SU_{R(G)}(C)/CU(M_{R(G)}(C)) \tag{e 2.15}$$

and $\Pi \circ J_{c(G)} = \mathrm{id}_G$.

It is important to note that, if $x \in SU_{R(G)}(C)$ and $[x] \in G \setminus \{0\}$ in $K_1(C)$. Then $J_c([x]) = \bar{x}$. In fact, since $[x] \in G_b \oplus \mathrm{Tor}(G)$, if $J_c([x]) = \bar{y}$, then $y \in SU_{R(G)}(C)$. It follows that $x^*y \in SU_{R(G)}(C) \cap U_0(M_{R(G)}(C)) \subset CU(C)$. So $\bar{y} = \bar{x}$. This fact will be also used without further notice. Note also that if $\mathrm{dim X} < \infty$, then we can let $R(K_1(C(X))) = \mathrm{dim}X$.

Therefore one obtains the following:

Proposition 2.16. *Let X, G, G_b and Π be as described in 2.15. Then there is an injective homomorphism $J_{c(G)} : G \to U_{R(G)}(C)/CU(M_{R(G)}(C))$ for some integer $R(G) \geq 1$ such that $\Pi \circ J_{c(G)} = \mathrm{id}_G$ and $J_{c(G)}(G_b \oplus Tor(G)) \subset SU_{R(G)}(C)/CU(M_{R(G)}(C))$. In what follows, we may write J_c instead of $J_{c(G)}$, if G is understood.*

Corollary 2.17. *Let X, G and G_b be as in 2.15 and let Y be a compact metric space. Suppose that $B = P_1M_r(C(Y))P_1$, where $P_1 \in M_r(C(Y))$ is a projection and $\varphi : C \to B$ is a unital homomorphism. Suppose also that $z_b \in G_b \oplus Tor(G)$ and $\varphi_{*1}(z_b) = 0$. Then $\varphi^{\ddagger}(J_{c(G)}(z_b)) = \bar{1}$ in $U(M_N(C))/CU(M_N(C))$ for some integer $N \geq 1$, when* $\mathrm{dim}Y = d < \infty$, *$N$ can be chosen to be* $\max\{R(G), d\}$.

Proof. Suppose that $u_b \in U(M_{R(G)}(C))$ such that $\bar{u} = J_{c(G)}(z_b)$. Without loss of generality, one may assume that $\varphi(u_b) \in U_0(M_{R(G)}(B))$, since $\varphi_{*1}(z_b) = 0$. By 2.16, $u_b \in SU_N(C)$. It follows from 2.10 that $\varphi(u_b) \in SU_{R(G)}(B)$. Thus, by 2.11,

$$\varphi(u_b) \in SU_{R(G)}(B) \cap U_0(M_{R(G)}(B)) \subset CU(M_{R(G)}(B)).$$

It follows that $\varphi^{\ddagger}(J_{c(G)}(z_b)) = \bar{1}$.

□

Definition 2.18. Let A be a unital C^*-algebra and let $u \in U_0(A)$. Denote by $\mathrm{cel}(u)$ the infimum of the length of the paths of unitaries of $U_0(A)$ which connects u with 1_A.

Definition 2.19. We say $(\delta, \mathcal{G}, \mathcal{P})$ is a KL-triple, if, for any δ-$\mathcal{G}$-multiplicative contractive completely positive linear map $L : A \to B$ (for any unital C^*-algebra B) $[L]|_{\mathcal{P}}$ is well defined. Moreover, if L_1 and L_2 are two δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps $L_1, L_2 : A \to B$ such that

$$\|L_1(g) - L_2(g)\| < \delta \text{ for all } g \in \mathcal{G}, \tag{e 2.16}$$

$$[L_1]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}}.$$

If $K_i(C)$ is finitely generated ($i = 0, 1$) and $\mathcal{P}$ is large enough, then $[L]|_{\mathcal{P}}$ defines an element in $KK(C, A)$ (see 2.4 of [31]). In such cases, we will write $[L]$ instead of $[L]|_{\mathcal{P}}$, and $(\delta, \mathcal{G}, \mathcal{P})$ is called a KK-triple and $(\delta, \mathcal{G})$ a KK-pair.

Now we also assume that A is amenable (or B is amenable). Let $u \in U(B)$ be such that $\|[L(g),\, u]\| < \delta_0$ for all $g \in \mathcal{G}_0$ for some finite subset $\mathcal{G}_0 \subset A$ and for some $\delta_0 > 0$. Then, we may assume that there exists contractive completely positive linear map $\Psi : A \otimes C(\mathbb{T}) \to B$ such that

$$\|L(g) - \Psi(g \otimes 1)\| < \delta \text{ for all } g \in \mathcal{G} \text{ and } \|\Psi(1 \otimes z) - u\| < \delta$$

(see 2.8 of [31]). Thus, we may assume that $\mathrm{Bott}(L,\, u)|_{\mathcal{P}}$ is well defined (see 2.10 in [31]). In what follows, when we say $(\delta, \mathcal{G}, \mathcal{P})$ is a KL-triple, we further assume that $\mathrm{Bott}(L, u)|_{\mathcal{P}}$ is well defined, provided that L is δ-$\mathcal{G}$-multiplicative and $\|[L(g),\, u]\| < \delta$ for all $g \in \mathcal{G}$. In case that $K_i(A)$ is finitely generated ($i = 0, 1$), we may even assume that $\mathrm{Bott}(L,\, u)$ is well defined. We also refer to 2.10 and 2.11 of [31] for $\mathrm{bott}_0(L, u)$ and $\mathrm{bott}_1(L, u)$. If u and v are unitary and $\|[u,\, v]\| < \delta$, we use $\mathrm{bott}_1(u, v)$ as in 2.10 and 2.11 of [31]. Let p is a projection and $\|[p,\, v]\| < \delta$, we may also write $\mathrm{bott}_0(p, v)$ for the element in $K_1(B)$ represented by a unitary which is close to $(1 - p) + pvp$.

Definition 2.20. If u is a unitary, we write $\langle L(u) \rangle = L(u)(L(u)^*L(u))^{-1/2}$ when $\|L(u^*)L(u) - 1\| < 1$ and $\|L(u)L(u^*) - 1\| < 1$. In what follows we will always assume that $\|L(u^*)L(u) - 1\| < 1$ and $\|L(u)L(u^*) - 1\| < 1$, when we write $\langle L(u) \rangle$.

Let B be another unital C^*-algebra and let $\varphi : A \to B$ be a unital homomorphism. Then $\langle \varphi \circ L(u) \rangle = \varphi(\langle L(u) \rangle)$. Let $u \in CU(A)$. Then, for any $\epsilon > 0$, if δ is sufficiently small and $\mathcal{G}$ is sufficiently large (depending on u) and L is δ-$\mathcal{G}$-multiplicative, then

$$\mathrm{dist}(\langle L(u) \rangle, CU(B)) < \epsilon.$$

Let $\delta > 0$, $\mathcal{G} \subset A$ be a finite subset, $\mathcal{W} \subset U(A)$ be a finite subset and $\epsilon > 0$. We say $(\delta, \mathcal{G}, \mathcal{W}, \epsilon)$ is a $\mathcal{U}$-quadruple, provided the following hold: if for any δ-$\mathcal{G}$-multiplicative contractive completely positive linear map $L : A \to B$, $\langle L(y) \rangle$ is well defined,

$$\|\langle L(u) \rangle - L(u)\| < \epsilon/2 \text{ and } \|\langle L(u) \rangle - \langle L(v) \rangle\| < \epsilon/2,$$

if $u,\, v \in \mathcal{U}$ and $\|u - v\| < \delta$. We also require that, if $u \in CU(A) \cap \mathcal{U}$,

$$\|\langle L(u) \rangle - c\| < \epsilon/2$$

for some $c \in CU(B)$. We make one additional requirement. Let $G_{\mathcal{U}}$ be the subgroup of $U(A)/CU(A)$ generated by $\{\bar{u} : u \in \mathcal{U}\}$. There exists a homomorphism $\lambda : U_{\mathcal{U}} \to U(B)$ such that

$$\mathrm{dist}(\overline{\langle L(u) \rangle}, \lambda(u)) < \epsilon \text{ for all } u \in \mathcal{U}$$

(see Appendix 14.5 for a proof that such λ exists). We may denote $L^{\ddagger}$ for a fixed homomorphism λ. Note that, when $\epsilon < 1$, $[\langle L(u) \rangle] = \Pi_c(L^{\ddagger}(\bar{u}))$ in $K_1(B)$, where $\Pi_c : U(B)/CU(B) \to K_1(B)$ is the induced homomorphism.

2.21. Let A and B be two unital C^*-algebra.. Suppose that A is a separable amenable C^*-algebra. Let $\mathcal{Q} \subset K_0(A)$ be a finite subset. Then $\boldsymbol{\beta}(Q) \subset K_1(A \otimes C(\mathbb{T}))$. Let $\mathcal{W}$ be a finite subset of $U(M_n(A \otimes C(\mathbb{T})))$ such that its image in $K_1(A \otimes C(\mathbb{T}))$ containing $\boldsymbol{\beta}(Q)$. Denote by $G(\mathcal{Q})$ the subgroup generated by Q. Fix $\epsilon > 0$. Let $(\delta, \mathcal{G}, \mathcal{W}, \epsilon)$ be a $\mathcal{U}$-quadruple. Let $J' : \boldsymbol{\beta}(G(\mathcal{Q})) \to U(M_N(A \otimes C(\mathbb{T})))/CU(M_N(A \otimes C(\mathbb{T})))$ be defined in 2.15. Let $L : A \to B$

be a δ-$\mathcal{G}$-multiplicative contractive completely positive linear map and let $u \in U(B)$ such that $\|[L(g), u]\| < \delta$ for all $g \in \mathcal{G}$. With sufficiently small δ and large $\mathcal{G}$, let Ψ be given in 2.19, we may assume that $\Psi^{\ddagger}$ is defined on $J'(\boldsymbol{\beta}(G(\mathcal{Q})))$. We denote this map by

$$\mathrm{Bu}(\varphi, u)(x) = \Psi^{\ddagger}(J'(x))) \text{ for all } x \in \mathcal{Q}. \tag{e 2.17}$$

We may assume that $[p_1], [p_2], ..., [p_k]$ generates $\mathcal{Q}$, where $p_1, p_2, ..., p_k$ are assumed to be projections in $M_N(A)$. Let $z_j = (1 - p_j) + p_j(1 \otimes z)^{(N)}$ (see 2.3), $j = 1, 2, ..., k$. Then z_j is a unitary in $M_N(A \otimes C(\mathbb{T}))$. Suppose that $A = C(X)$ for some compact metric space. In the above, we let $J' = J_{c(\boldsymbol{\beta}(G(\mathcal{Q}))} = J_c$ and $N = R(\boldsymbol{\beta}(G(\mathcal{Q})))$. Note $z_j \notin SU_N(C(X) \otimes C(\mathbb{T}))$. If $[p_i] - [p_j] \in \mathrm{ker}\rho_{C(X)}$, then for each $x \in X$, there is a unitary $w \in M_N$ such that $w^* p_i(x) w = p_j$. Then $\det(z_i z_j^*(x)) = 1$. In other words, $z_i z_j^* \in SU_N(C(X) \otimes C(\mathbb{T}))$. Note, by the end of 2.15, $J_c([p_j]) = \bar{z}_j$, $j = 1, 2, ..., k$. If B has stable rank d, we may assume that, $R(\boldsymbol{\beta}(G(\mathcal{Q}))) \geq d + 1$. In what follows, when we write $\mathrm{Bu}(\varphi, u)(x)$, or $\mathrm{Bu}(\varphi, u)|_{\mathcal{Q}}$, we mean that δ is sufficiently small and $\mathcal{G}$ is sufficiently large so that $L^{\ddagger}$ is well defined on $J_c(\boldsymbol{\beta}(x)$, or on $J_c(\boldsymbol{\beta}(\mathcal{Q}))$. Moreover, we note that $[L]|_{\mathcal{Q}} = \Pi' \circ L^{\ddagger}|_{\boldsymbol{\beta}(\mathcal{Q})}$. Furthermore, by choosing even smaller δ, we may also assume that when

$$\|[\varphi(g), u]\| < \delta \text{ and } \|[\varphi(g), v]\| < \delta \text{ for all } g \in \mathcal{G}$$

$$\mathrm{dist}(\mathrm{Bu}(\varphi, uv)(x), \mathrm{Bu}(\varphi, u)(g) + \mathrm{Bu}(\varphi, v)(x)) < \epsilon \text{ for all } x \in \mathcal{Q}.$$

Lemma 2.22. *Let X be a connected finite CW complex of dimension $d > 0$. Then $K_*(C(X^{(d-1)})) = G_{0,*} \oplus K_*(C(X))/F_d K_*(C(X))$, where $G_{0,*}$ is a finitely generated free group. Consequently,*

$$K_1(C(X^{(1)})) = S \oplus K_1(C(X))/F_3 K_1(C(X)),$$

where S is a finitely generated free group.

Proof. Let $I^{(d)} = \{f \in C(X) : f|_{X^{(d-1)}} = 0\}$. Then $I^{(d)} = C_0(Y)$, where $Y = S^d \vee S^d \vee \cdots \vee S^d$. In particular, $K_*(I^{(d)})$ is free. Therefore, by applying 4.1 of [17],

$$K_i(C(X^{(d-1)})) \cong G_{0,i} \oplus K_i(C(X))/F_d K_i(C(X)),$$

where $G_{0,i}$ is isomorphic to a subgroup of $K_{i-1}(I^{(d)})$, $i = 0, 1$.

For the last part of the lemma, we use the induction on d. It obvious holds for $d = 1$. Assume the last part holds for $\dim X = d \geq 1$. Suppose that $\dim X = d + 1$. Let $s_m : C(X) \to C(X^{(m)}), s_1 : C(X) \to C(X^{(1)}), s_{m,1} : C(X^{(m)}) \to C(X^{(1)})$ be defined by the restrictions $(0 < m \leq d)$. We have that $s_1 = s_{m,1} \circ s_m$. Note that if Y is a finite CW complex with dimension 2, then $K_1(\mathrm{ker} s_1^Y) = \{0\}$, where $s_1^Y : C(Y) \to C(Y^{(1)})$ is the surjective map induced by the restriction, where $Y^{(1)}$ is the 1-skeleton of Y. It follows that the induced map $(s_1^Y)_{*1}$ from $K_1(C(Y))$ into $K_1(Y^{(1)})$ is injective. This fact will be used in the following computation.

We have

$$K_1(C(X^{(d)})) = S_0 \oplus K_1(C(X))/F_d K_1(C(X)) = S_0 \oplus (s_d)_{*1}(K_1(C(X))) \tag{e 2.18}$$

and, by 4.1 of [17],

$$\begin{aligned}
K_1(C(X^{(1)}) &= S_1 \oplus K_1(C(X^{(d)})/F_3 K_1(C(X^{(d)})) && \text{(e 2.19)}\\
&= S_1 \oplus (s_{d,1})_{*1}(K_1(C(X^{(d)})) && \text{(e 2.20)}\\
&= S_1 \oplus (s_{d,1})_{*1}(S_0 \oplus (s_d)_{*1}(K_1(C(X)))) && \text{(e 2.21)}\\
&= S_1 \oplus (s_{d,1})_{*1}(S_0) \oplus (s_{d,1})_{*1}((s_d)_{*1}(K_1(C(X)))) && \text{(e 2.22)}\\
&= S_1 \oplus (s_{d,1})_{*1}(S_0) \oplus (s_1)_{*1}(K_1(C(X))) && \text{(e 2.23)}\\
&= S_1 \oplus (s_{d,1})_{*1}(S_0) \oplus (s_{2,1})_{*1} \circ (s_2)_{*1}(K_1(C(X))) && \text{(e 2.24)}\\
&= S_1 \oplus (s_{d,1})_{*1}(S_0) \oplus (s_2)_{*1}(K_1(C(X))/F_3 K_1(C(X))) && \text{(e 2.25)}\\
&= S_1 \oplus (s_{d,1})_{*1}(S_0) \oplus K_1(C(X))/F_3 K_1(C(X))). && \text{(e 2.26)}
\end{aligned}$$

Put $S = S_1 \oplus (s_{d,1})_{*1}(S_0)$. Note $K_1(C(X^{(1)})$ is free. So S must be a finitely generated free group. This ends the induction.

□

3 Definition of $\mathcal{C}_1$

Definition 3.1. Let A be a C^*-algebra and let $a, b \in A_+$. Recall that we write $a \lesssim b$ if there exists a sequence $x_n \in A_+$ such that

$$\lim_{n\to\infty} \|x_n^* b x_n - a\| = 0.$$

If $a = p$ is a projection and $a \lesssim b$, there is a projection $q \in Her(b)$ and a partial isometry $v \in A$ such that $vv^* = p$ and $v^*v = q$.

Definition 3.2. Let $0 < d < 1$. Define $f_d \in C_0((0, \infty])$ by $f_d(t) = 0$ if $t \in [0, d/2]$, $f_d(t) = 1$ if $t \in [d, \infty)$, and f_d is linear in $(d/2, d)$.

Definition 3.3. Denote by $\mathcal{I}^{(0)}$ the class of finite dimensional C^*-algebras and denote by $\mathcal{I}^{(1,0)}$ the class of C^*-algebras with the form $C([0,1]) \otimes F$, where $F \in \mathcal{I}^{(0)}$. For an integer $k \geq 1$, denote by $\mathcal{I}^{(k)}$ the class of C^*-subalgebra with the form $PM_r(C(X))P$, where $r \geq 1$ is an integer, X is a finite CW complex of covering dimension at most k and $P \in M_r(C(X))$ is a projection.

Definition 3.4. Denote by $\mathcal{I}_k$ the class of those C^*-algebras which are quotients of C^*-algebras in $\mathcal{I}^{(k)}$. Let $C \in \mathcal{I}_k$. Then $C = PM_r(C(X))P$, where X is a compact subset of a finite CW complex, $r \geq 1$ and $P \in M_r(C(X))$ is a projection. Furthermore, there exists a finite CW complex Y of dimension k such that X is a compact subset of Y and there is a projection $Q \in M_r(C(Y))$ such that $\pi(Q) = P$, where $\pi : M_r(C(Y)) \to M_r(C(X))$ is the quotient map defined by $\pi(f) = f|_X$.

Definition 3.5. Let A be a unital C^*-algebra. We say that A is a locally AH-algebra, if for any finite subset $\mathcal{F} \subset A$ and any $\epsilon > 0$, there exists a C^*-subalgebra $C \in \mathcal{I}_k$ (for some $k \geq 0$) such that

$$\text{dist}(a, C) < \epsilon \ \text{ for all } \ a \in \mathcal{F}.$$

A is said to be locally AH-algebra with no dimension growth, if there exists an integer $d \geq 0$, for any finite subset $\mathcal{F} \subset A$, any $\epsilon > 0$ and any $\eta > 0$, there exists a C^*-subalgebra $C \subset A$ with the form $C = PM_r(C(X))P \in \mathcal{I}_d$ such that

$$\text{dist}(a, C) < \epsilon \ \text{ for all } \ a \in \mathcal{F}. \tag{e 3.27}$$

Definition 3.6. Let $g : \mathbb{N} \to \mathbb{N}$ be a nondecreasing map. Let A be a unital simple C^*-algebra. We say that A is in $\mathcal{C}_g$ if the following holds: For any finite subset $\mathcal{F} \subset A$, any $\epsilon > 0$ and any $a \in A_+ \setminus \{0\}$, there is a projection $p \in A$ and a C^*-subalgebra $C = PM_r(C(X))P \in \mathcal{I}_d$ with $1_C = p$ such that

$$\|pa - ap\| < \epsilon \ \text{ for all } \ a \in \mathcal{F}, \tag{e 3.28}$$
$$\text{dist}(pap, C) < \epsilon \ \text{ for all } \ a \in \mathcal{F}, \tag{e 3.29}$$
$$\frac{d+1}{\text{rank}(P(x))} < \frac{\eta}{g(d)+1} \ \text{ for all } \ x \in X \ \text{ and} \tag{e 3.30}$$
$$1 - p \lesssim a. \tag{e 3.31}$$

If $g(d) = d$ for all $d \in \mathbb{N}$, we say $A \in \mathcal{C}_1$.

Let $\mathcal{B}$ be a class of unital C^*-algebras. Let A be a unital simple C^*-algebra. We say that A is tracially in $\mathcal{B}$ if the following holds: For any finite subset $\mathcal{F} \subset A$, any $\epsilon > 0$, any $a \in A_+ \setminus \{0\}$ and any $\eta > 0$, there is a projection $p \in A$ and a C^*-subalgebra $B \in \mathcal{B}$ with $1_B = p$ such that

$$\begin{aligned} \|pa - ap\| &< \epsilon \text{ for all } a \in \mathcal{F}, && \text{(e 3.32)} \\ \mathrm{dist}(pap, B) &< \epsilon \text{ for all } a \in \mathcal{F} \text{ and} && \text{(e 3.33)} \\ 1 - p &\lesssim a. && \text{(e 3.34)} \end{aligned}$$

We write $A \in \mathcal{C}_{g,\infty}$, if A is tracially $\mathcal{I}_d$ for some integer $d \geq 0$.

Using the fact that A is unital simple, it is easy to see that if $A \in \mathcal{C}_{g,\infty}$ then $A \in \mathcal{C}_g$. Moreover, $\mathcal{C}_g \subset \mathcal{C}_1$ for all g.

4 C^*-algebras in $\mathcal{C}_g$

Proposition 4.1. *Every unital hereditary C^*-subalgebra of a unital simple C^*-algebra in $\mathcal{C}_g$ (or in $\mathcal{C}_{g,\infty}$) is in $\mathcal{C}_g$(or is in $\mathcal{C}_{g,\infty}$).*

Proof. Let A be a unital simple C^*-algebra in $\mathcal{C}_g$ and let $e \in A$ be a non-zero projection. Let $B = eAe$ be a unital hereditary C^*-subalgebra of A. To prove that B is in $\mathcal{C}_g$, let $\mathcal{F} \subset B$ be a finite subset, let $1 > \epsilon > 0$, $b \in B_+ \setminus \{0\}$ and let $\eta > 0$.

Since A is simple, there are $x_1, x_2, ..., x_m \in A$ such that

$$\sum_{i=1}^{m} x_i^* e x_i = 1_A. \qquad \text{(e 4.35)}$$

Let $\mathcal{F}_1 = \{e\} \cup \mathcal{F} \cup \{x_1, x_2, ..., x_m, x_1^*, x_2^*, ..., x_m^*\}$. Put $M = \max\{\|a\| : a \in \mathcal{F}_1\}$. Since $A \in \mathcal{C}_g$, there is a projection $p \in A$, a C^*-subalgebra $C \subset A$ with $1_C = p$, $C = PM_r(C(X))P$, where X is a compact subset of a finite CW complex with dimension d, $r \geq 1$ is an integer and $P \in M_r(C(X))$ is a projection, such that

$$\begin{aligned} \frac{d+1}{\mathrm{rank} P(\xi)} &< (\eta/16(m+1))(1/(g(d)+1)) \text{ for all } \xi \in X, && \text{(e 4.36)} \\ \|px - xp\| &< \frac{\epsilon}{256(m+1)M} \text{ for all } x \in \mathcal{F}, && \text{(e 4.37)} \\ \mathrm{dist}(pxp, C) &< \frac{\epsilon}{256(m+1)M} \text{ for all } x \in \mathcal{F} \text{ and} && \text{(e 4.38)} \\ 1 - p &\lesssim b. && \text{(e 4.39)} \end{aligned}$$

There is a projection $q_1 \in C$, a projection $q_2 \in B$ and $y_1, y_2, ..., y_m \in C$ such that

$$\|q_1 - pep\| < \frac{\epsilon}{128(m+1)M}, \qquad \text{(e 4.40)}$$

$$\|q_2 - epe\| < \frac{\epsilon}{128(m+1)M} \text{ and } \sum_{i=1}^{m} y_i^* q_1 y_i = p. \qquad \text{(e 4.41)}$$

It follows that $q_1(x)$ has rank at least rank $P(x)/m$ for each $x \in X$. One also has

$$\|q_1 - q_2\| < \frac{\epsilon}{64(m+1)M}. \qquad \text{(e 4.42)}$$

There is a unitary $w \in A$ such that

$$\|w - 1\| < \frac{\epsilon}{32(m+1)M} \text{ and } w^* q_1 w = q_2. \qquad \text{(e 4.43)}$$

Define $C_1 = w^* q_1 C q_1 w$. Then $C_1 \cong q_1 C q_1 \in \mathcal{I}_d$. Moreover,

$$\frac{d+1}{\mathrm{rank}(w^* q_1 w)(x)} < \eta/(g(d)+1) \ \text{ for all } \ x \in X. \tag{e 4.44}$$

For any $a \in \mathcal{F}$,

$$\begin{aligned} \|q_2 a - a q_2\| &\le 2\|q_1 - q_2\|\|a\| + \|q_1 eae - eaeq_1\| & \text{(e 4.45)} \\ &< \epsilon. & \text{(e 4.46)} \end{aligned}$$

Moreover, if $c \in C$ such that

$$\|pap - c\| < \frac{\epsilon}{256(m+1)M}, \tag{e 4.47}$$

then

$$\begin{aligned} \|q_1 a q_1 - q_1 c q_1\| &\le \|q_1 a q_1 - pepapep\| + \|pepapep - q_1 c q_1\| & \text{(e 4.48)} \\ &< \frac{2\epsilon}{256(m+1)} + \frac{2\epsilon}{256(m+1)} = \frac{\epsilon}{64(m+1)}. & \text{(e 4.49)} \end{aligned}$$

It follows from (e 4.40), (e 4.42), (e 4.43) and (e 4.49) that

$$\begin{aligned} \|q_2 a q_2 - w^* q_1 c q_1 w\| &= \|q_2 a q_2 - w^* q_2 w a w^* q_2 w\| & \text{(e 4.50)} \\ &+ \|w^* q_2 w a w^* q_2 w - w^* q_1 c q_1 w\| & \text{(e 4.51)} \\ &< \frac{4\epsilon}{32(m+1)M} + \|q_2 w a w^* q_2 - q_1 c q_1\| & \text{(e 4.52)} \\ &< \frac{\epsilon}{8(m+1)} + \frac{2\epsilon}{32(m+1)} + \|q_2 a q_2 - q_1 c q_1\| & \text{(e 4.53)} \\ &< \frac{6\epsilon}{32(m+1)} + \frac{2\epsilon}{64(m+1)} + \|q_1 a q_1 - q_1 c q_1\| & \text{(e 4.54)} \\ &< \frac{7\epsilon}{32(m+1)} + \frac{\epsilon}{64(m+1)} = \frac{15\epsilon}{64(m+1)}. & \text{(e 4.55)} \end{aligned}$$

Therefore, for all $a \in \mathcal{F}$,

$$\mathrm{dist}(q_2 a q_2, C_1) < \epsilon. \tag{e 4.56}$$

Note that

$$\begin{aligned} \|(e - q_2) - (e - q_2)(1-p)(e - q_2)\| &\le \|(e - q_2) - e(1-p)e\| & \text{(e 4.57)} \\ &\le \|q_2 - epe\| < \frac{\epsilon}{128(m+1)M}. & \text{(e 4.58)} \end{aligned}$$

So, in particular, $(e - q_2)(1-p)(e - q_2)$ is invertible in $(e - q_2)A(e - q_2)$. Therefore $[e - q_2] = [(e - q_2)(1-p)(e - q_2)]$ in the Cuntz equivalence. But

$$\begin{aligned} (e - q_2)(1-p)(e - q_2) &\lesssim (1-p)(e - q_2)(1-p) \lesssim (1-p) & \text{(e 4.59)} \\ &\lesssim b. & \text{(e 4.60)} \end{aligned}$$

We conclude that, in B,

$$(e - q_2) \lesssim b. \tag{e 4.61}$$

It follows from (e 4.44), (e 4.45), (e 4.56) and (e 4.61) that B is in $\mathcal{C}_g$.

Since $C_1 \in \mathcal{I}_d$, if we assume that A is in $\mathcal{C}_{g,\infty}$. Then the above shows that $B \in \mathcal{C}_{g,\infty}$.

□

Proposition 4.2. *A unital simple C^*-algebra A which satisfies condition* (e 3.28), (e 3.29) *and* (e 3.30) *in the definition of 3.6 has property (SP), i.e., every non-zero hereditary C^*-subalgebra B of A contains a non-zero projection.*

Proof. Let A be a unital simple C^*-algebra satisfies the condition (e 3.28), (e 3.29) and (e 3.30) in 3.6. Suppose that $B \subset A$ is a hereditary C^*-subalgebra. Choose $a \in B_+ \setminus \{0\}$ with $\|a\| = 1$. Choose $1/4 > \lambda > 0$. Let $f_1 \in C([0,1])$ be such that $0 \le f_1 \le 1$, $f_1(t) = 1$ for all $t \in [1-\lambda/4, 1]$ and $f_1(t) = 0$ for $t \in [0, 1-\lambda/2]$. Put $b = f_1(a)$. Let $f_2 \in C([0,1])$ be such that $0 \le f_2 \le 1$, $f_2(t) = 1$ for all $t \in [1-\lambda/2, 1]$ and $f_2(t) = 0$ for $t \in [0, 1-\lambda]$. Put $c = f_2(a)$. Then $b \ne 0$, $bc = b$ and $b, c \in B$.

Since A is simple, there are $x_1, x_2, ..., x_m \in A$ such that

$$\sum_{i=1}^m x_i^* b x_i = 1. \qquad \text{(e 4.62)}$$

Let $M = \max\{\|x_i\| : 1 \le i \le m\}$.

Let $\mathcal{F} = \{a, b, c, x_1, x_2, ..., x_m, x_1^*, x_2^*, ..., x_m^*\}$. For any $1/16 > \epsilon > 0$, there is a projection $p \in A$ and a C^*-subalgebra $C \subset A$ with $1_C = p$, where $C = PM_r(C(X))P$, $r \ge 1$ is an integer, X is a compact subset of a finite CW complex with dimension $d(X)$, $P \in M_r(C(X))$ is a projection such that

$$\begin{aligned} \frac{d(X)+1}{\mathrm{rank}P(x)} &< 1/4m, && \text{(e 4.63)}\\ \|pf - fp\| &< \epsilon/64(m+1)(M+1) \text{ for all } f \in \mathcal{F}, && \text{(e 4.64)}\\ \text{and } \mathrm{dist}(pfp, B) &< \epsilon/64(m+1)(M+1). && \text{(e 4.65)} \end{aligned}$$

There is $a_1 \in C$ such that

$$\|pap - a_1\| < \epsilon/64(m+1)(M+1). \qquad \text{(e 4.66)}$$

By choosing sufficiently small ϵ, we may assume that

$$\|f_1(a_1) - pf_1(a)p\| < 1/64, \quad \|f_2(a_1) - pf_2(a)p\| < 1/64, \qquad \text{(e 4.67)}$$

and there are $y_1, y_2, ..., y_m \in C$ such that

$$\|\sum_{i=1}^m y_i^* f_1(pap) y_i - p\| < 1/16 \text{ and } \|\sum_{i=1}^m y_i^* f_1(a) y_i - p\| < 1/16. \qquad \text{(e 4.68)}$$

We may also assume that

$$\begin{aligned} \|f_2(pap) - f_2(a_1)\| &< 1/64 && \text{(e 4.69)}\\ \|f_1(pap) - f_1(a_1)\| &< 1/64. && \text{(e 4.70)} \end{aligned}$$

Let q be the open projection which is given by $\lim_{k\to\infty}(f_1(a_1))^{1/k}$. Then by (e 4.68), $q(x)$ has rank at least $\mathrm{rank}P(x)/m$. Thus, in $M_r(C(X))$, $f_1(a_1)(x)$ has rank at least $4d(X)$ for all $x \in X$, by (e 4.63). It follows from Proposition 3.2 of [13] that there is a non-zero projection $e \in \overline{f_1(a_1)Cf_1(a_1)}$. Note that $f_2(a_1)f_1(a_1) = f_1(a_1)$. Therefore

$$f_2(a_1)e = e. \qquad \text{(e 4.71)}$$

It follows from that

$$\begin{aligned}\|ce-e\| &\le \|pcpe-e\|=\|pf_2(a)pe-e\| &\text{(e 4.72)}\\ &\le \|pf_2(a)pe-f_2(a_1)e\|+\|f_2(a_1)e-e\|<1/64. &\text{(e 4.73)}\\ & &\text{(e 4.74)}\end{aligned}$$

Similarly,

$$\|ec-e\|<1/16. \tag{e 4.75}$$

It follows from 2.5.4 of [23] that there is a projection $e_1\in\overline{cAc}\subset B$ such that

$$\|e-e_1\|<1.$$

□

Proposition 4.3. *Let A be a unital simple C^*-algebra in $\mathcal{C}_g$ (or in $\mathcal{C}_{g,\infty}$). Then in* (e 3.29), *we can also assume that, for any $\epsilon>0$,*

$$\|pxp\|\ge\|x\|-\epsilon \text{ for all } x\in\mathcal{F}.$$

Proof. The proof of this is contained in that of 4.2. Note that in the proof of 4.2, (e 4.68) implies that $f_1(pap)\neq 0$. This implies that $\|pap\|\ge 1-\lambda/2$. With $\lambda=\epsilon$, this will gives $\|pap\|\ge 1-\lambda$. This holds for any finitely many given positive elements with $\|a\|=1$. If $0\neq\|a\|\neq 1$, it is clear that, by considering elements $a/\|a\|$, we can have $\|pap\|\ge\|a\|-\epsilon$. In general, when x is not positive, we can enlarge the finite subset $\mathcal{F}$ so it also contains x^*x. We omit the full proof.

□

Proposition 4.4. *Let A be a unital simple C^*-algebra. Then the following are equivalent:*
(1) *$A\in\mathcal{C}_g$ (or in $\mathcal{C}_{g,\infty}$);*
(2) *for any integer $n\ge 1$, $M_n(A)\in\mathcal{C}_g$ (or in $\mathcal{C}_{g,\infty}$),*
(3) *for some integer $n\ge 1$, $M_n(A)\in\mathcal{C}_g$ (or in $\mathcal{C}_{g,\infty}$).*

Proof. It is clear that (2) implies (3). That (3) implies (1) follows from 4.1. To prove (1) implies (2), let $n\ge 1$ be an integer, let $\mathcal{F}\subset M_n(A)$ be a finite subset, $\epsilon>0$, $a\in M_n(A)_+\setminus\{0\}$ and $\sigma>0$. We first consider the case that $A\in\mathcal{C}_1$.

Since A is simple, so is $M_n(A)$. Let $\{e_{i,j}\}$ be a matrix unit for M_n. We identify A with $e_{11}Ae_{11}$. Therefore, (by 3.3.4 of [23], for example), there is a non-zero element $a_1\in A_+$ such that $a_1\lesssim a$. Since A has property (SP), by 3.5.7 of [23], there are n mutually orthogonal and mutually equivalent nonzero projections $q_1,q_2,...,q_n\in\overline{a_1Aa_1}$. Choose a finite subset $\mathcal{F}_1\subset A$ such that

$$\{(a_{i,j})_{n\times n}: a_{i,j}\in\mathcal{F}_1\}\supset\mathcal{F}. \tag{e 4.76}$$

Since we assume that $A\in\mathcal{C}_1$, there exists a projection $q\in A$, a C^*-subalgebra $C\subset A$ with $1_C=q$, $C=PM_r(C(X))P\in\mathcal{I}_k$, where $r\ge 1$ is an integer, X is a compact subset of a finite CW complex with dimension $d(X)$ and $P\in M_r(C(X))$ is a projection, such that

$$\begin{aligned}\|qa-aq\| &< \epsilon/n^2 \text{ for all } a\in\mathcal{F}_1 &\text{(e 4.77)}\\ \mathrm{dist}(pap,C) &< \epsilon/n^2 \text{ for all } a\in\mathcal{F}_1, &\text{(e 4.78)}\\ \frac{d(X)+1}{\mathrm{rank}P(x)} &< \sigma/(g(d)+1) \text{ for all } x\in X \text{ and} &\text{(e 4.79)}\\ 1_A-p &\lesssim q_1. &\text{(e 4.80)}\end{aligned}$$

Now let $p = \mathrm{diag}(\overbrace{q, q, ..., q}^{n})$, $Q = \mathrm{diag}(\overbrace{P, P, ..., P}^{n})$ and $B = M_n(C)$. Then $B = QM_{nr}(C(X))Q$. Moreover

$$\begin{aligned} \|pa - ap\| &< \epsilon \text{ for all } a \in \mathcal{F}, && \text{(e 4.81)} \\ \mathrm{dist}(pap, B) &< \epsilon \text{ for all } a \in \mathcal{F}, && \text{(e 4.82)} \\ \frac{d(X)+1}{\mathrm{rank} Q(x)} &< \sigma/n(g(d)+1) \text{ for all } x \in X. && \text{(e 4.83)} \end{aligned}$$

Furthermore,

$$1_{M_n(A)} - q \lesssim \mathrm{diag}(q_1, q_2, ..., q_n) \lesssim a_1 \lesssim a. \quad \text{(e 4.84)}$$

Thus $M_n(A) \in \mathcal{C}_1$.

For the case that $A \in \mathcal{C}_{g,1}$, the proof is the same with obvious modification. We omit the details.

□

The following is certainly known.

Lemma 4.5. *Let X be a compact subset of a finite CW complex, $r \geq 1$ be an integer and let $E \in M_r(C(X))$ be a projection. Then, there is a projection $Q \in M_k(EM_r(C(X))E)$ for some integer $k \geq 1$ such that $QM_k(EM_r(C(X))E)Q \cong M_l(C(X))$ and there is a unitary $W \in M_k(EM_r(C(X))E)$ such that $W^*EW \leq Q$.*

Proof. There is a decreasing sequence of finite CW complexes $\{X_n\}$ of dimension d (for some integer $d \geq 1$) such that $X = \cap_{n=1}^{\infty} X_n$. There is an integer $n \geq 1$ and a projection $E' \in M_r(C(X_n))$ such that $E'|_X = E$. There is an integer $k \geq 1$ and a projection $Q' \in M_k(E'M_r(C(X_n))E')$ which is unitarily equivalent to the constant projection id_{M_l} for some $l \geq \mathrm{rank}E' + d$. Note that there is a unitary $W' \in M_k(E'M_r(C(X_n))E')$ such that

$$(W')^*E'W' \leq Q'.$$

Let $W = W'|_X$. Then $W^*EW \leq Q$. Let $Z \in M_k(E'M_r(C(X_n))E')$ be a unitary such that

$$Z^*Q'Z = \mathrm{id}_{M_l}.$$

Let $Q = Q'|_X$. Then $QM_k(EM_r(C(X))E)Q \cong M_l(C(X))$.

□

Lemma 4.6. *Let $C = PM_r(C(X))P$, where $r \geq 1$ is an integer, X is a compact subset set of a finite CW complex and $P \in M_r(C(X))$ is a projection. Suppose that A is a unital C^*-algebra with the property (SP) and suppose that $\varphi : C \to A$ is a unital homomorphism. Then, for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists a non-zero projection $p \in A$ and a finite dimensional C^*-subalgebra $B \subset A$ with $1_B = p$ such that*

$$\begin{aligned} \|p\varphi(f) - \varphi(f)p\| &< \epsilon \textit{ for all } f \in \mathcal{F}, && \text{(e 4.85)} \\ \mathrm{dist}(\varphi(f), B) &< \epsilon && \text{(e 4.86)} \\ \text{and } \|p\varphi(f)p\| &\geq \|\varphi(f)\| - \epsilon \textit{ for all } x \in \mathcal{F}. && \text{(e 4.87)} \end{aligned}$$

Proof. We first prove the case that $C = M_r(C(X))$. It is clear that this case can be reduced further to the case that $C = C(X)$. Denote by $C_1 = \varphi(C(X)) \cong C(Y)$, where $Y \subset X$ is a compact subset. Let $\epsilon > 0$ and let $\mathcal{F} = \{f_1, f_2, ..., f_m\} \subset C$. There are $x_1, x_2, ..., x_m \in X$ such that

$$|\varphi(f_i)(x_i)| = \|\varphi(f_i)\|, \;\; i = 1, 2, ..., m. \tag{e 4.88}$$

There exists $\delta > 0$ such that

$$|f_i(x) - f_i(x_i)| < \epsilon/2 \tag{e 4.89}$$

for all x for which $\mathrm{dist}(x, x_i) < \delta$, $i = 1, 2, ..., m$. Let $g_i \in C(X)$ defined by $g_i(x) = 1$ if $\mathrm{dist}(x, x_i) < \delta/4$ and $g_i(x) = 0$ if $\mathrm{dist}(x, x_i) \geq \delta/2$, $i = 1, 2, ..., m$. For each i, by 4.2, there exists a non-zero projection $e_i \in \overline{\varphi(g_i)A\varphi(g_i)}$, $i = 1, 2, ..., m$. Put $p = \sum_{i=1}^m e_i$. Let B be the C^*-subalgebra generated by $e_1, e_2, ..., e_m$. Then B is isomorphic to a direct sum of m copies of $\mathbb{C}$. In particular, B is of dimension m. As in Lemma 2 of [20], this implies that

$$\|p\varphi(f) - \varphi(f)p\| < \epsilon \text{ and} \tag{e 4.90}$$

$$\|p\varphi(f)p - \sum_{j=1}^m f(x_j)e_j\| < \epsilon \tag{e 4.91}$$

for all $f \in \mathcal{F}$. By (e 4.88),

$$\|p\varphi(f_i)p\| \;\; \geq \;\; \|\sum_{j=1}^m f_i(x_j)e_j\| - \epsilon \tag{e 4.92}$$

$$\geq \;\; |f_i(x_i)| - \epsilon = \|\varphi(f_i)\| - \epsilon, \tag{e 4.93}$$

$i = 1, 2, ..., m$. This prove the case that $C = C(X)$. For the case that $C = M_r(C(X))$, we note that $\varphi(e_{11})A\varphi(e_{11})$ is simple and has (SP), where $\{e_{i,j}\}$ is a matrix unit for M_r. Thus this case follows from the case that $C = C(X)$. Note that, in this case $\dim B = r^2 m$.

Now we consider the general case. There is an integer $K \geq 1$ and a projection $Q \in M_K(C)$ such that $QM_K(C)Q \cong M_r(C(X))$. By choosing a projection with larger rank, we may assume that $\mathrm{rank} Q \geq \mathrm{rank} P + 2dim(X)$. By conjugating a unitary, we may further assume that $Q \geq 1_C$. Define $\Phi = \varphi \otimes \mathrm{id}_{M_K} : M_K(C) \to M_K(A)$ and define $\Psi : QM_K(C)Q \to A_1 = \Phi(Q)M_K(A)\Phi(Q)$ by $\Psi = \Phi|_{QM_K(C)Q}$.

Now let $\epsilon > 0$ and let $\mathcal{F} \subset C$ be a finite subset. Let $\mathcal{F}_1 = \{Q, 1_C\} \cup \{Q((a_{i,j})_{K\times K})Q : a_{i,j} \in \mathcal{F}\}$. Let $M = \max\{\|f\| : f \in \mathcal{F}_1\}$. From what we have shown, there is a projection $p_1 \in M_K(A)$ and a finite dimensional C^*-subalgebra $B_1 \subset M_K(A)$ with $1_{B_1} = p_1$ such that

$$\|p_1\Phi(f) - \Phi(f)p_1\| \;\; < \;\; \epsilon/8(M+1) \tag{e 4.94}$$

$$\mathrm{dist}(\Phi(f), B_1) \;\; < \;\; \epsilon/8(M+1) \text{ and} \tag{e 4.95}$$

$$\|p_1\Phi(f)p_1\| \;\; \geq \;\; \|\Phi(f)\| - \epsilon/8(M+1) \tag{e 4.96}$$

for all $f \in \mathcal{F}_1$. There is a projection $e \in B_1$ such that

$$\|p_1\Phi(1_C)p_1 - e\| < \epsilon/4(M+1). \tag{e 4.97}$$

Then, for $f \in \mathcal{F} \subset C$,

$$\|e\varphi(f) - \varphi(f)e\| \;\; \leq \;\; 2M\|e - p_1\Phi(1_C)p_1\| + \|p_1\Phi(1_C)p_1\Phi(f) - \Phi(f)p_1\Phi(1_C)p_1\| \tag{e 4.98}$$

$$< \;\; \epsilon/4 + \epsilon/8(M+1) + \|p_1\Phi(1_C)\Phi(f) - \Phi(f)\Phi(1_C)p_1\| \tag{e 4.99}$$

$$< \;\; \epsilon/4 + \epsilon/8(M+1) + \|p_1\Phi(f) - \Phi(f)p_1\| \tag{e 4.100}$$

$$< \;\; \epsilon/4 + \epsilon/8(M+1) + \epsilon/8(M+1) < \epsilon. \tag{e 4.101}$$

Let $B = eB_1e$. Then we also have

$$\mathrm{dist}(e\varphi(f)e, B) = \mathrm{dist}(e\varphi(f)e, eB_1e) < \epsilon/8(M+1) < \epsilon \ \text{ for all } \ f \in \mathcal{F}. \tag{e 4.102}$$

It follows from (e 4.96) that

$$\begin{aligned} \|e\varphi(f)e\| &\geq \|p_1\Phi(1_C)p_1\Phi(f)p_1\Phi(1_C)p_1\| - \epsilon/8(M+1) & \text{(e 4.103)}\\ &\geq \|p_1\Phi(1_C)\Phi(f)\Phi(1_C)p_1\| - \epsilon/4(M+1) & \text{(e 4.104)}\\ &= \|p_1\Phi(f)p_1\| - \epsilon/2(M+1) & \text{(e 4.105)}\\ &\geq \|\Phi(f)\| - \epsilon/(M+1) = \|\varphi(f)\| - \epsilon/(M+1) & \text{(e 4.106)} \end{aligned}$$

for all $f \in \mathcal{F}$. This completes the proof. □

Proposition 4.7. *Let A be a unital simple C^*-algebra with the property (SP). Suppose that A satisfies the following conditions: For any $\delta > 0$ and any finite subset $\mathcal{G} \subset A$, there exits a projection $q \in A$, a C^*-subalgebra $C \in \mathcal{I}_k$ for some $k \geq 1$ with $1_C = p$ such that*

$$\begin{aligned} \|qx - xq\| &< \delta, & \text{(e 4.107)}\\ \mathrm{dist}(qxq, C) &< \epsilon \ \text{and} & \text{(e 4.108)}\\ \|qxq\| &\geq \|x\| - \epsilon \ \textit{for all} \ x \in \mathcal{G}. & \text{(e 4.109)} \end{aligned}$$

Then A satisfies the Popa condition: for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there is a projection $p \in A$ and a finite dimensional C^*-subalgebra $B \subset A$ with $1_B = p$ such that

$$\|px - xp\| < \epsilon, \ \mathrm{dist}(pxp, B) < \epsilon \ \text{ and } \ \|pxp\| \geq \|x\| - \epsilon \tag{e 4.110}$$

for all $x \in \mathcal{F}$.

Proof. Let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. By the assumption, there is a projection q and a C^*-subalgebra $C \subset A$ with $C \in \mathcal{I}_k$ and with $1_C = q$ such that

$$\|qx - xq\| < \epsilon/8, \ \mathrm{dist}(qxq, C) < \epsilon/8 \ \text{ and } \ \|qxq\| \geq \|x\| - \epsilon \tag{e 4.111}$$

for all $x \in \mathcal{F}$. For each $x \in \mathcal{F}$, let $c_x \in C$ such that

$$\|pxp - c_x\| < \epsilon/8. \tag{e 4.112}$$

It follows from 4.6 that, there is $p \in qAq$ and a finite dimensional C^*-subalgebra $B \subset qAq$ with $1_B = p$ such that

$$\begin{aligned} \|pc_x - c_xp\| &< \epsilon/8, & \text{(e 4.113)}\\ \mathrm{dist}(pc_xp, B) &< \epsilon/8 \ \text{and} & \text{(e 4.114)}\\ \|pc_xp\| &\geq \|c_x\| - \epsilon/8 & \text{(e 4.115)} \end{aligned}$$

for all $x \in \mathcal{F}$. It follows that

$$\begin{aligned} \|px - xp\| &\leq \|pqx - pc_x\| + \|c_xp - xqp\| & \text{(e 4.116)}\\ &< \epsilon/8 + \|pqxq - pc_x\| + \epsilon/8 + \|c_xp - qxqp\| & \text{(e 4.117)}\\ &< \epsilon/8 + \epsilon/8 + \epsilon/8 + \epsilon/8 < \epsilon & \text{(e 4.118)} \end{aligned}$$

for all $x \in \mathcal{F}$. Also

$$\begin{aligned} \mathrm{dist}(pxp, B) &\leq \|pxp - pc_xp\| + \mathrm{dist}(pc_xp, B) & \text{(e 4.119)}\\ &= \|pqxqp - pc_xp\| + \epsilon/8 < \epsilon & \text{(e 4.120)} \end{aligned}$$

for all $x \in \mathcal{F}$. Moreover,

$$\begin{aligned} \|pxp\| &= \|pqxqp\| \geq \|pc_xp\| - \epsilon/8 & \text{(e 4.121)} \\ &\geq \|c_x\| - \epsilon/4 \geq \|qxq\| - \epsilon/4 - \epsilon/8 & \text{(e 4.122)} \\ &\geq \|x\| - \epsilon/8 - \epsilon/4 - \epsilon/8 \geq \|x\| - \epsilon & \text{(e 4.123)} \end{aligned}$$

for all $x \in \mathcal{F}$.

□

Corollary 4.8. *Let A be a unital simple C^*-algebra in $\mathcal{C}_1$. Then A satisfies the Popa condition.*

Theorem 4.9. *Let A be a unital separable simple C^*-algebra in $\mathcal{C}_1$. Then A is MF, quasi-diagonal and $T(A) \neq \emptyset$.*

Proof. By 4.2 of [29], every unital separable C^*-subalgebra which satisfies the Popa condition is MF ([3]). To see A is quasi-diagonal, let $\epsilon > 0$ and let $\mathcal{F} \subset A$ be a finite subset. Let $\mathcal{F}_1 = \{a, b, ab : a, b \in \mathcal{F}\}$. Since A has the Popa condition, there is a non-zero projection $e \in A$ and a finite dimensional C^*-subalgebra $B \subset A$ with $1_B = e$ such that $\|ex - xe\| < \epsilon/2$ for all $x \in \mathcal{F}_1$, $\|exe\| > \|exe\| - \epsilon$ and $\mathrm{dist}(x, B) < \epsilon$ for all $x \in \mathcal{F}_1$. There is a contractive completely positive linear map $L : eAe \to B$ such that $\|L(exe) - exe\| < \epsilon$ for all $x \in \mathcal{F}_1$. Define $L_1 : A \to B$ by $L_1(x) = L(exe)$ for all $x \in A$. Then $\|L_1(x)\| \geq \|exe\| - \epsilon$ for all $x \in \mathcal{F}_1$ and $\|L_1(a)L_1(b) - L_1(ab)\| < \epsilon$ for all $a, b \in \mathcal{F}$. It follows from Theorem 1 of [57] that A is quasi-diagonal. Since it is MF, it has tracial state. It follows that A is finite. □

5 Regularity of C^*-algebras in $\mathcal{C}_1$

The following lemma is a variation of 3.3 of [13].

Lemma 5.1. *Let X be a compact metric space with covering dimension $d \geq 0$. Let $r \geq 1$ be an integer and $P, p \in M_r(C(X))$ be non-zero projections such that $p \leq P$. Suppose that $\mathrm{rank}\, P - \mathrm{rank}\, p$ is at least $3(d+1)$ at each $x \in X$ and $a \in pM_r(C(X))p$. Then, for any $\epsilon > 0$, there exists an invertible element $x \in PM_r(C(X))P$ such that*

$$\|a - x\| < \epsilon.$$

Proof. We first consider the case that X is a finite CW complex with dimension d. In this case, by consider each summand separately, without loss of generality, we may assume that X is connected. Let $A = PM_r(C(X))P$ and let $k = \mathrm{rank\,p}$. Let $q \in M_N(C(X))$ for some large N such that $\mathrm{rank\,q} = k + d + 1$ and q is trivial. By 6.10.3 of [1], p is unitarily equivalent to a subprojection of q. Thus, we find a projection $q_1 \in M_N(C(X))$ with rank $d+1$ such that $p \oplus q_1$ is trivial. Since $\mathrm{rank}(P - p)$ is at least $3(d+1)$, by 6.10.3 of [1] , there exists a trivial projection $p_1 \in (P-p)M_r(C(X))(P-p)$ with rank $2d+1$ and q_1 is unitarily equivalent to a subprojection of p_1. Therefore, we may assume that $q_1 \in (P-p)M_r(C(X))(P-p)$.

Put $B = (p + q_1)A(p + q_1)$. Then $B \cong M_{k+d+1}(C(X))$. Note that $P - (p + q_1)$ has rank at least $2d + 2$. As above, find a trivial projection $q_2 \in (P - (p + q_1))A(P - (p + q_1))$ with rank at least $d + 1$. Then, by 3.3 of [13], there is an invertible element $z \in (p + q_1 + q_2)A(p + q_1 + q_2)$ such that

$$\|a - z\| < \epsilon/2.$$

Define $x = z + \epsilon/2(P - (p + q_1 + q_1))$. Then x is invertible in A and $\|a - x\| < \epsilon$. This proves the case that X is a finite CW complex. In general, there exists a sequence finite CW complexes X_n with dimension at most d such that $C(X) = \lim_{n\to\infty} C(X_n)$ and $M_r(C(X)) = \lim_{n\to\infty} M_r(C(X_n))$.

Let $\varphi_n : M_r(C(X_n)) \to M_r(C(X))$ be the homomorphism induced by the inductive limit system. We may assume that φ_n is unital. There is, for some $n \geq 1$, $Q, p' \in M_r(C(X_n))$ such that

$$\|\varphi_n(Q) - P\| < \epsilon/32, \ \ \|\varphi_n(p') - p\| < \epsilon/32 \ \text{ and } \ p' \leq Q$$

(see the proof of 2.7.2 of [23]). Moreover there is a unitary $U \in M_r(C(X_n))$ such that $\|U - 1\| < \epsilon/16$ such that

$$U^*\varphi(Q)U = P.$$

We may also assume that there is $b \in QM_r(C(X_n))Q$ such that

$$\|\varphi_n(b) - a\| < \epsilon/16.$$

Let $b' = p'bp' \in QM_r(C(X_n))Q$. Then

$$\|\varphi_n(b') - a\| < \epsilon/8.$$

By what we have proved, there is an invertible element $c \in QM_r(C(X_n))Q$ such that

$$\|b' - c\| < \epsilon/4.$$

Note that A is a unital hereditary C^*-subalgebra of $M_r(C(X))$. Put $x = U^*\varphi_n(b')U$. Then x is an invertible element in A. We also have

$$\|a - x\| < \epsilon.$$

□

Theorem 5.2. *Let A be a unital simple C^*-algebra in $\mathcal{C}_1$. Then A has stable rank one.*

Proof. We may assume that A is infinite dimensional. Let $a \in A$ be a nonzero element. We will show that a is a norm limit of invertible elements. So we may assume that a is not invertible and $\|a\| = 1$. Since A is finite (4.9), a is not one-sided invertible. Let $\epsilon > 0$. By 3.2 of [49], there is a zero divisor $b \in A$ such that

$$\|a - b\| < \epsilon/2. \tag{e 5.124}$$

We may further assume that $\|b\| \leq 1$. Therefore, by [49], there is a unitary $u \in A$ such that ub is orthogonal to a non-zero positive element $x \in A$. Put $d = ub$. Since A has (SP) (by 4.2), there exists $e \in A$ such that $de = ed = 0$. Since A is also simple (for example, by 3.5.7 of [23]), we may write $e = e_1 \oplus e_2$ with $e_2 \lesssim e_1$.

Note that $d, e_2 \in (1-e_1)A(1-e_1)$ and $(1-e_1)A(1-e_1) \in \mathcal{C}_1$ (by 4.1). Since $(1-e_1)A(1-e_1)$ is simple, there are $x_1, x_2, ..., x_m \in (1-e_1)A(1-e_1)$ such that

$$\sum_{i=1}^m x_i^* e_2 x_i = 1 - e_1. \tag{e 5.125}$$

Let $\delta > 0$. Let $\mathcal{F} = \{d, e_2, x_1, x_2, ..., x_m, x_1^*, x_2^*, ..., x_m^*\}$. There exists a projection $p \in (1-e_1)A(1-e_1)$ and a unital C^*-subalgebra $C \subset A$ with $1_C = p$ and with $C = PM_r(C(X))P$, where $r \geq 1$ is an integer, X is a compact subset of a finite CW complex with dimension $d(X)$ and $P \in M_r(C(X))$ is a projection such that

$$\|px - xp\| < \delta \text{ for all } x \in \mathcal{F}, \tag{e 5.126}$$

$$\mathrm{dist}(pxp, C) < \delta \text{ for all } x \in \mathcal{F}, \tag{e 5.127}$$

$$\frac{d(X)+1}{\mathrm{rank}P(t)} < 1/8(m+1) \text{ for all } t \in X \text{ and} \tag{e 5.128}$$

$$(1 - e_1 - p) \lesssim e_2 \lesssim e_1. \tag{e 5.129}$$

With sufficiently small δ, we may assume that

$$\|e_2 - (e_2' + e_2'')\| < \epsilon/16 \text{ and } \|d - (d_1 + d_2)\| < \epsilon/16, \tag{e 5.130}$$

where $e_2' \in C$ and $e_2'' \in (1 - e_1 - p)A(1 - e_1 - p)$ are nonzero projections, $d_1 \in (p - e_2')C(p - e_2')$ and $d_2 \in (1 - e_1 - p)A(1 - e_1 - p)$. Moreover, there are $y_1, y_2, ..., y_m \in C$ such that

$$\|\sum_{i=1}^{m} y_i^* e_2' y_i - p\| < \epsilon/16. \tag{e 5.131}$$

By (e 5.129), there is a partial isometry $v \in A$ such that $v^*v = 1 - e_1 - p$ and $vv^* \leq e_1$. Put

$$e_1' = vv^* \text{ and } d_2' = (\epsilon/8)(e_1 - e_1') + (\epsilon/8)v + (\epsilon/8)v^* + d_2.$$

Then $(\epsilon/8)v + (\epsilon/8)v^* + d_2$ has a matrix decomposition in $(e_1' + (1 - e_1 - p))A(e_1' + (1 - e_1 - p))$:

$$\begin{pmatrix} 0 & \epsilon/8 \\ \epsilon/8 & d_2 \end{pmatrix}.$$

It follows that d_2' is invertible in $(1 - p)A(1 - p)$. Note also

$$\|d_2 - d_2'\| < \epsilon/8. \tag{e 5.132}$$

In C, we have $d_1 e_2' = e_2' d_1 = 0$. It follows from (e 5.131) and (e 5.128) that e_2' has rank at least

$$\text{rankP}(x)/m \geq 8d(X) + 1. \tag{e 5.133}$$

It follows from 5.1 that there exists an invertible element $d_1' \in C$ such that

$$\|d_1 - d_1'\| < \epsilon/16.$$

Therefore $d' = d_1' + d_2'$ is invertible in A. However,

$$\begin{aligned} \|d - (d_1' + d_2')\| &\leq \|(d - (d_1 + d_2)\| + \|d_1 - d_1'\| + \|d_2 - d_2'\| && \text{(e 5.134)} \\ &< \epsilon/16 + \epsilon/16 + \epsilon/8 = \epsilon/4. && \text{(e 5.135)} \end{aligned}$$

Thus

$$\|b - u^*d'\| = \|u^*u(b - u^*d')\| = \|ub - d'\| = \|d - d'\| < \epsilon/4. \tag{e 5.136}$$

Finally

$$\|a - u^*d'\| \leq \|a - b\| + \|b - u^*d'\| < \epsilon/2 + \epsilon/4 < \epsilon \tag{e 5.137}$$

Note that u^*d' is invertible. □

Some version of the following is known. The relation of ϵ and $\epsilon^2/2^9$ in the statement is not sharp but will be needed in the proof of 5.5.

Lemma 5.3. *Let A be a C^*-algebra, $a \in A_+$ with $0 \leq a \leq 1$ and let $p \in A$ be a projection. Let $1 > \epsilon > 0$. Then*

$$f_\epsilon(a) \lesssim f_{\epsilon^2/2^9}(pap + (1 - p)a(1 - p)). \tag{e 5.138}$$

Proof. We will work in $M_2(A)$ and identify A with $e_{11}M_2(A)e_{11}$, where $\{e_{i,j}\}_{1\le i,j\le 2}$ is a matrix unit for M_2.

Let $x = \begin{pmatrix} pa^{1/2} & 0 \\ (1-p)a^{1/2} & 0 \end{pmatrix}$. Then,

$$x^*x = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \text{ and } xx^* = \begin{pmatrix} pap & pa(1-p) \\ (1-p)ap & (1-p)a(1-p) \end{pmatrix}. \tag{e 5.139}$$

We compute that

$$\|(1-f_{\epsilon^2/2^9}(pap))pap\| < \epsilon^2/2^8, \tag{e 5.140}$$
$$\|(1-f_{\epsilon^2/2^9}((1-p)a(1-p)))(1-p)a(1-p)-(1-p)a(1-p)\| < \epsilon^2/2^8. \tag{e 5.141}$$

Moreover,

$$\|(1-f_{\epsilon^2/2^9}(pap))pa^{1/2}\|^2 = \|(1-f_{\epsilon^2/2^9}(pap))pap(1-f_{\epsilon^2/2^9}(pap))\| < \epsilon^2/2^8. \tag{e 5.142}$$

Therefore

$$\|(1-f_{\epsilon^2/2^9}(pap))pa(1-p)\| < \epsilon/2^4. \tag{e 5.143}$$

Similarly

$$\|(1-f_{\epsilon^2/2^9}((1-p)a(1-p))(1-p)ap\| < \epsilon/2^4. \tag{e 5.144}$$

Put $b = \mathrm{diag}(pap, (1-p)a(1-p))$. Then

$$\|xx^* - f_{\epsilon^2/2^9}(b)xx^*f_{\epsilon^2/64}(b)\| < \epsilon/2 \tag{e 5.145}$$

It follows from Lemma 2.2 of [50] that

$$f_\epsilon(xx^*) \quad \lesssim \quad f_{\epsilon^2/2^9}(b)xx^*f_{\epsilon^2/2^9}(b) \le f_{\epsilon^2/2^9}(b) \tag{e 5.146}$$
$$\tag{e 5.147}$$

This implies that, in A,

$$f_\epsilon(a) \lesssim f_{\epsilon^2/2^9}(pap + (1-p)a(1-p)).$$

□

The following is a standard compactness fact. We include here for convenience.

Lemma 5.4. *Let X be a compact metric space and $g \in C(X)$ with $0 \le g(x) \le 1$ for all $x \in X$. Suppose that f_0 is a lower-semi continuous function on X such that $1 \ge f_0(x) > g(x)$ for all $x \in X$. Suppose also that $f_n \in C(X)$ is a sequence of continuous functions with $f_n(x) > 0$ for all $x \in X$, $f_n(x) \le f_{n+1}(x)$ for all x and n, and $\lim_{n\to\infty} f_n(x) = f_0(x)$ for all $x \in X$. Then there is $N \ge 1$ such that*

$$f_n(x) > g(x) \text{ for all } x \in X \tag{e 5.148}$$

and for all $n \ge N$.

Proof. For each $x \in X$, there exists $N(x) \geq 1$ such that

$$f_n(x) > g(x) \text{ for all } n \geq N(x). \tag{e 5.149}$$

Since $f_{N(x)} - g$ is continuous, there is $\delta(x) > 0$ such that

$$f_{N(x)}(y) > g(y) \text{ for all } y \in B(x, \delta(x)). \tag{e 5.150}$$

Then $\cup_{x \in X} B(x, \delta(x)) \supset X$. There are $x_1, x_2, ..., x_m \in X$ such that $\cup_{i=1}^m B(x_i, \delta(x_i)) \supset X$. Let $N = \max\{N(x_1), N(x_2), ..., N(x_m)\}$. Then, if $n \geq N$, for any $x \in X$, there exists i such that $x \in B(x_i, \delta(x_i))$. Then

$$f_n(x) \geq f_{N(x_i)}(x) \geq g(x). \tag{e 5.151}$$

□

Theorem 5.5. *If A is a unital simple C^*-algebra in $\mathcal{C}_1$, then A has the strict comparison for positive elements in the following sense: If $a, b \in A_+$ and*

$$d_\tau(a) < d_\tau(b) \text{ for all } \tau \in T(A),$$

then $a \lesssim b$, where $d_\tau(a) = \lim_{\epsilon \to 0} \tau(f_\epsilon(a))$.

.

Proof. Let a, $b \in A_+$ be two non-zero elements such that

$$d_\tau(a) < d_\tau(b) \text{ for all } \tau \in T(A). \tag{e 5.152}$$

For convenience, we assume that $\|a\|, \|b\| = 1$. Let $1/2 > \epsilon > 0$. Put $c = f_{\epsilon/16}(a)$. If c is Cuntz equivalent to a, then zero is an isolated point in $sp(a)$. So, a is Cuntz equivalent to a projection. Then $d_\tau(a)$ is continuous on $T(A)$. Since $d_\tau(b)$ is lower-semi continuous on $T(A)$, the inequality (e 5.152) implies that

$$r_0 = \inf\{d_\tau(b) - d_\tau(c) : \tau \in T(A)\} > 0. \tag{e 5.153}$$

Otherwise, there is a nonzero element $c' \in \overline{aAa}_+$ such that $c'c = cc' = 0$. Therefore

$$\inf\{d_\tau(a) - d_\tau(c) : \tau \in T(A)\} > 0. \tag{e 5.154}$$

So in either way, (e 5.153) holds.

Put $c_1 = f_{\epsilon/64}(a)$. It follows from 5.4 that there is $1 > \delta_1 > 0$ such that

$$\tau(f_{\delta_1}(b)) > \tau(c) \geq d_\tau(c_1) \text{ for all } \tau \in T(A). \tag{e 5.155}$$

Put $b_1 = f_{\delta_1}(b)$. Then

$$r = \inf\{\tau(b_1) - d_\tau(c_1) : \tau \in T(A)\} \geq \inf\{\tau(b_1) - \tau(c) : \tau \in T(A)\} > 0. \tag{e 5.156}$$

Note that $\|b\| = 1$. Since A is simple and has (SP) (by 4.2), there is a non-zero projection $e \in A$ such that $b_1 e = e$ and

$$\tau(e) < r/8 \text{ for all } \tau \in T(A). \tag{e 5.157}$$

Let $r_1 = \inf\{\tau(e) : \tau \in T(A)\}$. Note that, since A is simple and $T(A)$ is compact, $r_1 > 0$. Let $b_2 = (1-e)b_1(1-e)$. Thus, there is $0 < \delta_2 < \delta_1/2 < 1/2$ such that

$$7r/8 < \inf\{\tau(f_{\delta_2}(b_2) - \tau(c_1) : \tau \in T(A)\} < r - r_1. \tag{e 5.158}$$

Since $f_{\delta_2}(b_2)f_{3/4}(b_2)=f_{3/4}(b_2)$ and since $\overline{f_{3/4}(b_2)Af_{3/4}(b_2)}$ is non-zero, there is a non-zero projection $e_1\in A$ such that $e_1f_{\delta_2}(b_2)=e_1$ with $\tau(e_1)<r/18$ for all $\tau\in T(A)$.

There are $x_1,x_2,...,x_m\in A$ such that

$$\sum_{i=1}^m x_i^*e_1x_i=1. \tag{e 5.159}$$

Let $\sigma=\min\{\epsilon^2/2^{17}(m+1),\delta_1/8,r_1/2^7(m+1)\}$.

By [8], there are $z_1,z_2,...,z_K\in A$ and $b'\in A_+$ such that

$$\|f_{\epsilon^2/2^{12}}(c)-\sum_{j=1}^K z_j^*z_j\|<\sigma/4 \text{ and } \|f_{\delta_2}(b_2)-(b'+e_1+\sum_{j=1}^K z_jz_j^*\|<\sigma/4. \tag{e 5.160}$$

Let

$$\mathcal{F}=\{a,b,c,e,c_1,b_1,b',e_1\}\cup\{x_i,x_i^*:1\le i\le m\}\cup\{z_j,z_j^*:1\le j\le K\}.$$

Since $A\in\mathcal{C}_1$, for any $\eta>0$, there exists a projection $p\in A$, a C^*-subalgebra $C\subset A$ with $1_C=p$ and with the form $C=PM_r(C(X))P$, where $r\ge 1$ is an integer, X is a compact subset of a finite CW complex with dimension $d(X)$ and $P\in M_r(C(X))$ is a projection such that

$$\begin{aligned}
\|px-xp\| &< \eta \text{ for all } x\in\mathcal{F}, && \text{(e 5.161)}\\
\mathrm{dist}(pxp,C) &< \eta \text{ for all } x\in\mathcal{F} && \text{(e 5.162)}\\
\frac{d(X)+1}{\mathrm{rank}P(\xi)} &< 1/256(m+1) \text{ for all } \xi\in X \text{ and} && \text{(e 5.163)}\\
1-p &\lesssim e. && \text{(e 5.164)}
\end{aligned}$$

By choosing sufficiently small η, we obtain $b_3,c_2,b''\in C_+$, a projection $q_1\in C$, $y_1,y_2,...,y_m\in C$, $z_1',z_2',...,z_K\in C$ such that

$$\|pcp-c_2\|<\sigma,\ \|f_{\epsilon^2/2^{14}}(pcp)-f_{\epsilon^2/2^{14}}(c_2)\|<\sigma, \tag{e 5.165}$$

$$\|pb_2p-b_3\|<\sigma,\ \|f_{\delta_2}(pb_2p)-f_{\delta_1}(b_3)\|<\sigma,\ \|f_{\delta_2/4}(pb_2p)-f_{\delta_2/4}(b_3)\|<\sigma. \tag{e 5.166}$$

$$\|pe_1p-q_1\|<\sigma,\ \|\sum_{i=1}^m y_i^*q_1y_i-p\|<\sigma \tag{e 5.167}$$

and, (using (e 5.160)), such that

$$\|f_{\epsilon^2/2^{14}}(c_2)-\sum_{j=1}^K (z_j')^*z_j'\|<\sigma \text{ and} \tag{e 5.168}$$

$$\|f_{\delta_2}(b_3)-(\sum_{j=1}^K z_j'(z_j')^*+q_1+b'')\|<\sigma. \tag{e 5.169}$$

Note that, by (e 5.167) and (e 5.163),

$$\mathrm{rank}(q_1)(\xi)\ge\mathrm{rankP}(\xi)/m\ge 256(d(X)+1) \text{ for all } \xi\in X. \tag{e 5.170}$$

Therefore

$$t(q_1)\ge 1/m \text{ for all } t\in T(C). \tag{e 5.171}$$

It follows that

$$t(q_1)-2\sigma > 9(d(X)+1)/m \ \text{ for all } \ t\in T(C). \tag{e 5.172}$$

Therefore, by (e 5.168),

$$\begin{aligned} d_t(f_{\epsilon^2/2^{13}}(c_2))+9d(X)/m &\leq t(f_{\epsilon^2/2^{14}}(c_2))\leq \sigma+\sum_{j=1}^K t((z_j')^*z_j')+9d(X)/m && \text{(e 5.173)}\\ &= \sigma+9d(X)/m+\sum_{j=1}^K t(z_j'(z_j')^*) && \text{(e 5.174)}\\ &\leq (t(q_1)-\sigma)+\sum_{j=1}^K t(z_j'(z_j')^*) && \text{(e 5.175)}\\ &\leq t(f_{\delta_1}(b_3))\leq d_t(f_{\delta_1/2}(b_3)) && \text{(e 5.176)} \end{aligned}$$

for all $t\in T(C)$. It follows from by 3.15 of [53]

$$f_{\epsilon^2/2^{13}}(c_2)\lesssim f_{\delta_1/2}(b_2) \tag{e 5.177}$$

By (e 5.166) and Lemma 2.2 of [50],

$$f_{\delta_1/2}(b_3)\leq f_{\delta_1/4}(pb_2p)\leq pb_2p. \tag{e 5.178}$$

By (e 5.165),

$$f_{\epsilon^2/2^{11}}(pcp)\lesssim f_{\epsilon^2/2^{12}}(c_2)\leq pb_2p. \tag{e 5.179}$$

It follows from 5.3 and (e 5.179) that

$$\begin{aligned} f_{\epsilon/2}(c) &\lesssim f_{\epsilon^2/2^{11}}(pcp+(1-p)c(1-p))\lesssim f_{\epsilon^2/2^{11}}(pcp)\oplus(1-p) && \text{(e 5.180)}\\ &\lesssim pb_2p+e\lesssim b_2+e\lesssim b_1\lesssim b. && \text{(e 5.181)} \end{aligned}$$

We also have

$$f_\epsilon(a)\lesssim f_{\epsilon/2}(f_{\epsilon/16}(a))=f_{\epsilon/2}(c)\lesssim b. \tag{e 5.182}$$

Since this holds for all $1>\epsilon>0$, by 2.4 of [50], we conclude that

$$a\lesssim b.$$

□

Theorem 5.6. *If A is a unital separable simple C^*-algebra in $\mathcal{C}_1$, then $K_0(A)$ is weakly unperforated Riesz group.*

Proof. Note that, for each integer $n\geq 1$, by 4.4, $M_n(A)\in\mathcal{C}_1$. Suppose that $p,\,q\in M_n(A)$ are two projections such that

$$\tau(p)>\tau(q) \ \text{ for all } \ \tau\in T(A). \tag{e 5.183}$$

Then, by 5.5, $q\lesssim p$. Therefore, if $x\in K_0(A)$ with $nx>0$ for some integer $n\geq 1$, then one may write $x=[p]-[q]$ for some projections $p,q\in M_k(A)$ for some integer $k\geq 1$. The fact that $nx>0$ implies that

$$n(\tau(p)-\tau(q))>0 \ \text{ for all } \ \tau\in T(A) \tag{e 5.184}$$

which implies that

$$\tau(p) > \tau(q) \text{ for all } \tau \in T(A). \tag{e 5.185}$$

It follows that $[p] > [q]$. So $x > 0$. This shows that $K_0(A)$ is weakly unperforated.

To show that $K_0(A)$ is a Riesz group, let $q \le p$ be two projections in $M_n(A)$ such that $p = p_1 + p_2$, where p_1 and p_2 are two mutually orthogonal projections in $M_n(A)$. We need to show that there are projections $q_1, q_2 \in M_n(A)$ such that $q = q_1 + q_2$ and $[q_1] \le [p_1]$ and $[q_2] \le [p_2]$. Since, by 4.4, $M_n(A) \in \mathcal{C}_1$, to simplify the notation, we may assume that $p, q \in A$.

Since $q \le q$, we may assume that

$$\tau(p) > \tau(q) \text{ for all } \tau \in T(A). \tag{e 5.186}$$

Therefore, (since A is simple and has (SP)), we obtain two non-zero projections $p_{0,1} \le p_1$ and $p_{0,2} \le p_2$, and another non-zero projection $e_{00} \in A$ such that

$$\tau(p') > \tau(q) + \tau(e_{0,0}) \text{ for all } \tau \in T(A), \tag{e 5.187}$$

where $p' = p - p_{0,1} - p_{0,2}$. Put $p_1' = p_1 - p_{0,1}$ and $p_2' = p_2 - p_{0,2}$. So $p' = p_1' + p_2'$. From what has been proved, we have

$$q \oplus e_{00} \lesssim p' = p_1' + p_2'. \tag{e 5.188}$$

Let $v \in A$ such that

$$v^*v = p + e_{00} \text{ and } vv^* \le p_1' + p_2'. \tag{e 5.189}$$

There are

$$x_1, x_2, ..., x_{m_1}, y_1, y_2, ..., y_{m_2}, z_1, z_2, ..., z_{m_3} \in A$$

such that

$$\sum_{i=1}^{m_1} x_i^* e_{0,0} x_i = 1, \sum_{i=1}^{m_2} y_i^* p_1' y_i = 1 \text{ and } \sum_{i=1}^{m_3} z_i^* p_2' z_i = 1. \tag{e 5.190}$$

Let

$$\mathcal{F}_0 = \{x_i, y_j, z_k : 1 \le i \le m_1, 1 \le j \le m_2, 1 \le k \le m_3\}.$$

Define

$$\mathcal{F} = \{p, q, p_1', p_2', e_{00}, v, v^*\} \cup \mathcal{F}_0.$$

Fix $\eta > 0$. Since $A \in \mathcal{C}_1$, there exists a projection $e \in A$, a C^*-subalgebra $C = PM_r(C(X))P \in \mathcal{I}_k$ with $1_C = e$ such that

$$\begin{aligned} \|ex - xe\| &< \eta \text{ for all } x \in \mathcal{F}, && \text{(e 5.191)} \\ \mathrm{dist}(exe, C) &< \eta \text{ for all } x \in \mathcal{F}, && \text{(e 5.192)} \\ \frac{k+1}{\mathrm{rank} P(\xi)} &< \frac{1}{64(m_1 + m_2 + m_3 + 1)} \text{ for all } \xi \in X \text{ and} && \text{(e 5.193)} \\ 1 - e &\lesssim p_{0,1}. && \text{(e 5.194)} \end{aligned}$$

With sufficiently small η, we may assume that there exist projections $p'', p_1'', p_2'', q', e_{00}' \in C$, there are $x_i', y_j', z_k' \in C$ ($1 \le i \le m_1$, $1 \le j \le m_2$ and $1 \le k \le m_3$) and there are projections

$q'', p_0 \in (1-e)A(1-e)$ such that

$$\|ep'e - p''\| < 1/16, \ \ \|p_1'' - ep_1'e\| < 1/16, \tag{e 5.195}$$

$$\|p_2'' - ep_2'e\| < 1/16, \ \ \|q' - eqe\| < 1/16, \tag{e 5.196}$$

$$\|q'' - (1-e)q(1-e)\| < 1/16, \ \ \|p_0 - (1-e)p'(1-e)\| < 1/16, \tag{e 5.197}$$

$$p'' = p_1'' + p_2'', \ \text{ and } \ \|\sum_{i=1}^{m_1}(x_i')^* e_{00}' x_i' - e\| < 1/16, \tag{e 5.198}$$

$$\|\sum_{j=1}^{m_2}(y_j')^* p_1' y_j - e\| < 1/16 \ \text{ and } \ \|\sum_{k=1}^{m_3}(z_k)^* p_2' z_k' - e\| < 1/16. \tag{e 5.199}$$

Moreover,

$$q' \oplus e_{00} \lesssim p'' \ \text{ in } \ M_2(C). \tag{e 5.200}$$

Note that

$$\mathrm{rank}(e_{00})(x) \ge \mathrm{rank}(P(x))/m_1 \ge 64(k+1), \tag{e 5.201}$$

$$\mathrm{rank}(p_1'')(x) \ge 64(k+1) \ \text{ and } \ \mathrm{rank}(p_2'')(x) \ge 64(k+1) \tag{e 5.202}$$

for all $x \in X$. Suppose that X is the disjoint union of compact subsets $X_1, X_2, ..., X_N$ such that $\mathrm{rank}(e_{00})$, $\mathrm{rank}(p_1')$ and $\mathrm{rank}(p_2')$ are all constant on each X_i, $i = 1, 2, ..., N$. On each X_i, there are non-negative integers $m_{0,1}, m_{0,2}$, m_1, and m_2 such that

$$\mathrm{rank}(q') - k = m_{0,1} + m_{0,2}, \tag{e 5.203}$$

$$\mathrm{rank}(p_1'') = m_1', \ \ \mathrm{rank}(p_2'') = m_2', \tag{e 5.204}$$

$$m_1' - 10k > m_{0,1}, m_2' - 10k > m_{0,2}. \tag{e 5.205}$$

It follows from 6.10.3 of [1] that $q'|_{X_i}$ has a trivial subprojection $q_{1,i}' \le q'|_{X_i}$ such that $\mathrm{rank}(q_{1,i}') = m_{0,1}$. Thus, by 6.10.3 of [1],

$$q_{1,i}' \lesssim p_1''|_{X_i}. \tag{e 5.206}$$

Now

$$\mathrm{rank} q'|_{X_i} - q_{1,i}' + 9k < m_2 = \mathrm{rank}(p_2''|_{X_i}). \tag{e 5.207}$$

It follows from 6.10.3 of [1] again that

$$q'|_{X_i} - q_{1,i}' \lesssim p_2''|_{X_i}. \tag{e 5.208}$$

Define projections q_1', $q_2' \in C$ such that

$$q_1'|_{X_i} = q_{1,i}' \ \text{ and } \ q_2'|_{X_i} = q'|_{X_i} - q_{1,i}', \tag{e 5.209}$$

$i = 1, 2, ..., N$. Then

$$q' = q_1' + q_2' \ \text{ and } \ q_1' \lesssim p_1'' \ \text{ and } \ q_2' \lesssim p_2''. \tag{e 5.210}$$

Note, from (e 5.196), $p_1'' \lesssim p_1'$ and $p_2'' \lesssim p_2'$. We also have

$$q'' \le (1-e) \lesssim p_{0,1}. \tag{e 5.211}$$

Put $q_1''=q''+q_1'$. Then

$$q_1''\lesssim p_{0,1}+p_1'=p_1. \tag{e 5.212}$$

By (e 5.196) and (e 5.197), there exists a unitary $v\in A$ such that

$$v^*(q''+q')v=q. \tag{e 5.213}$$

Define

$$q_1=v^*(q''+q_1'')v \text{ and } q_2=v^*(q_2')v. \tag{e 5.214}$$

Then $q=q_1+q_2$. But we also have

$$q_1\lesssim q''+q_1''\lesssim p_1 \text{ and } q_2\lesssim q_2'\lesssim p_2'\le p_2. \tag{e 5.215}$$

This ends the proof.

□

Proposition 5.7. *Let A be a unital simple C^*-algebra in $\mathcal{C}_1$. Then, for any non-zero projections p and q, and any integer $n\ge 1$, there are mutually orthogonal projections $p_1,p_2,...,p_n,p_{n+1}\in pAp$ such that*

$$p=\sum_{i=1}^{n+1}p_i,\ \ [p_i]=[p_1],\ \ i=1,2,...,n,$$

$p_{n+1}\lesssim p_1$ *and* $p_{n+1}\lesssim q$.

Proof. There are $v_1,v_2,...,v_K\in M_K(A)$ and a projection $p'\in A$ such that

$$\sum_{i=1}^{K}v_i^*p'v_i=1 \text{ and } p'\le p. \tag{e 5.216}$$

Let $\eta=\inf\{\tau(q):\tau\in T(A)\}$. Choose an integer $m\ge 1$ such that $1/m<\eta/2$.

Let $1/2>\delta>0$ and $\mathcal{F}=\{p,p',v_i,v_i^*,1\le i\le K\}$. Since A is in $\mathcal{C}_1$, there is a projection $e\in A$ and a C^*-subalgebra $C\subset A$ with $1_C=e$ such that $C=PM_r(C(X))P$, where $r\ge 1$ is an integer, X is a compact subset of a finite CW complex with dimension k and $P\in M_r(C(X))$ is a projection, and

$$\|ex-xe\|<\delta,\ \ \mathrm{dist}(exe,C)<\delta \text{ for all } x\in\mathcal{F}, \tag{e 5.217}$$
$$\frac{k+1}{\mathrm{rank}(P(\xi))}<\frac{1}{4(n+1)(K+1)} \text{ for all } \xi\in X \text{ and} \tag{e 5.218}$$
$$\tau(1-e)<\min\{\eta/2,1/8(nmK+2)\} \text{ for all } \tau\in T(A). \tag{e 5.219}$$

With sufficiently small δ, we may assume that there are projections e', $e_1'\in C$ such that

$$\|epe-e'\|<1/16,\ \ \|(1-e)p(1-e)-(1-e')\|<1/16, \tag{e 5.220}$$
$$\|ep'e-e_1\|<1/16 \text{ and} \tag{e 5.221}$$
$$K\mathrm{rank}(e'(\xi))\ge \mathrm{rank}(P(\xi)) \text{ for all } \xi\in X \tag{e 5.222}$$

It follows from (e 5.218) that

$$\mathrm{rank}(e'(\xi))\ge 4m(n+1)(k+1) \text{ for all } \xi\in X. \tag{e 5.223}$$

There is a trivial projection $e'' \le e'$ in $PM_r(C(X))P$ such that

$$\mathrm{rank}(e''(\xi)) \ge (4m(n+1)-1)(k+1) \ \text{and}\ \mathrm{rank}(e'-e'')(\xi) \le k+1 \tag{e 5.224}$$

for all $\xi \in X$. It follows that there are mutually orthogonal and mutually equivalent projections $p_1', p_2', ..., p_n' \in C$ such that

$$\sum_{i=1}^{n} p_i' \le e'' \ \text{and}\ (n+1)[p_1'] \ge [e'']. \tag{e 5.225}$$

and $e'' - \sum_{i=1}^{n} p_i'$ has rank less than n. This implies that

$$\tau(p_1') > \tau(e'-e'') + \tau(1-e') \ \text{for all}\ \tau \in T(A). \tag{e 5.226}$$

Put $p_{n+1}' = e' - e'' + (1-e')$. Then

$$[p] = [\sum_{i=1}^{n+1} p_i']. \tag{e 5.227}$$

Therefore there are mutually orthogonal projections $p_1, p_2, ..., p_{n+1} \in eAe$ such that

$$p = \sum_{i=1}^{n+1} p_i \ \text{and}\ [p_i] = [p_i], \ \ i = 1, 2, ..., n+1. \tag{e 5.228}$$

Note that

$$[p_{n+1}'] \le [q] \ \text{and}\ [p_{n+1}'] \le [p_1].$$

□

6 Traces

Proposition 6.1. *Let A be a unital separable simple C^*-algebra in $\mathcal{C}_1$. For any positive numbers $\{R_n\}$ such that $\lim_{n\to\infty} R_n = \infty$, there exists a sequence of $C_n = P_nM_{r(n)}(C(X_n))P_n$, where $r(n) \ge 1$ is an integer, X_n is a finite CW complex with dimension $k(n)$ and $P_n \in M_{r(n)}(C(X_n))$ is a projection, a sequence of projections $p_n \in A$, a sequence of contractive completely positive linear maps $L_n : A \to C_n$ and a sequence of unital homomorphisms $h_n : C_n \to p_nAp_n$ such that*

$$\lim_{n\to\infty} \|a - [(1-p_n)a(1-p_n) + h_n \circ L_n(a)]\| = 0 \ \textit{for all}\ a \in A \tag{e 6.229}$$

$$\frac{k(n)+1}{\mathrm{rank}(P_n(x))} < \frac{1}{R_n} \ \textit{for all}\ x \in X, \tag{e 6.230}$$

$$\lim_{n\to\infty} \sup\{\tau(1-p_n) : \tau \in QT(A)\} = 0 \ \text{and} \tag{e 6.231}$$

$$\lim_{n\to\infty} \sup_{\tau\in QT(A)} |\tau(h_n \circ L_n(a)) - \tau(a)| = 0 \ \textit{for all}\ a \in A. \tag{e 6.232}$$

Proof. Let $\{\mathcal{F}_n\} \subset A$ be an increasing sequence of finite subsets whose union is dense in A. Since A is in $\mathcal{C}_1$, there exists a sequence of projections $e_n \in A$ and a sequence of $B_n \subset A$ with $1_{B_n} = e_n$ and $B_n = P_nM_{r_n}(C(X_n))P_n$, where $r_n \ge 1$ is an integer, X_n is a compact subset of a finite CW complex with dimension $k(n)$ and $P_n \in M_{r_n}(C(X_n))$ is a projection, such that

$$\|e_nx - xe_n\| < 1/2^{n+2} \ \text{for all}\ x \in \mathcal{F}_n, \tag{e 6.233}$$

$$\mathrm{dist}(e_nxe_n, B_n) < 1/2^{n+2} \ \text{for all}\ x \in \mathcal{F}_n, \tag{e 6.234}$$

$$\frac{k(n)+1}{\mathrm{rank}(P_n(\xi))} < 1/R_n \ \text{for all}\ \xi \in X \ \text{and} \tag{e 6.235}$$

$$\tau(1-e_n) < 1/2^{n+1} \ \text{for all}\ \tau \in QT(A). \tag{e 6.236}$$

For each $x \in \mathcal{F}_n$, let $y(x) \in B_n$ such that

$$\|e_n x e_n - y(x)\| < 1/2^{n+2}. \tag{e 6.237}$$

Since B_n is amenable, it follows from Theorem 2.3.13 of [23] that there exists a unital contractive completely positive linear map $\psi_n : e_n A e_n \to B_n$ such that

$$\|\psi_n(y(x)) - \mathrm{id}_{B_n}(y(x))\| < 1/2^{n+2} \ \text{ for all } \ x \in \mathcal{F}_n. \tag{e 6.238}$$

Put $\mathcal{G}_n = \{y(x) : x \in \mathcal{F}_n\}$. By Corollary 6.8 of [36], there exists $C_n \in \mathcal{I}^{(k(n))}$ with the form $C_n = Q_n M_r(C(Y_n))Q_n$, where Y_n is a finite CW complex,

$$\min\{\operatorname{rank} Q_n(y) : y \in Y_n\} = \min\{\operatorname{rank} P_n(x) : x \in X_n\}. \tag{e 6.239}$$

and a unital $1/2^{n+2}$-$\mathcal{G}_n$-multiplicative contractive completely positive linear map $\psi_n' : B_n \to C_n$ and a surjective homomorphism $h_n : C_n \to B_n$ such that

$$h_n \circ \psi_n' = \mathrm{id}_{B_n}. \tag{e 6.240}$$

Now define

$$L_n = \psi_n' \circ \psi_n. \tag{e 6.241}$$

It is ready to check that

$$\lim_{n\to\infty} \|L_n(ab) - L_n(a)L_n(b)\| = 0 \ \text{ for all } \ a, b \in A. \tag{e 6.242}$$

By (e 6.233) and (e 6.234),

$$\lim_{n\to\infty} \|x - (e_n x e_n + (1-e_n)x(1-e_n))\| = 0 \ \text{ for all } \ x \in A. \tag{e 6.243}$$

Since quasitraces are norm continuous (Corollary II 2.5 of [2]), by (e 6.236), it follows that

$$\begin{aligned}
\tau(x) &= \lim_{n\to\infty} \tau(e_n x e_n + (1-e_n)x(1-e_n)) && \text{(e 6.244)}\\
&= \lim_{n\to\infty} (\tau(e_n x e_n) + \tau((1-e_n)x(1-e_n))) && \text{(e 6.245)}\\
&= \lim_{n\to\infty} (\tau(e_n x e_n)) && \text{(e 6.246)}\\
&= \lim_{n\to\infty} \tau(h_n \circ L_n(x)) && \text{(e 6.247)}
\end{aligned}$$

for all $x \in A_+$ and all quasitraces $\tau \in QT(A)$. Also by (e 6.235) and (e 6.239),

$$\frac{k(n)+1}{\operatorname{rank} Q_n(y)} < 1/R_n \ \text{ for all } \ y \in Y_n.$$

□

Corollary 6.2. *Let A be a unital separable simple C^*-algebra in $\mathcal{C}_1$. Then, there exists a sequence of unital C^*-algebra $A_n \in \mathcal{I}^{k(n)}$, a unital sequence of contractive completely positive linear maps $L_n : A \to A_n$ and a sequence of unital homomorphisms $h_n : A_n \to A$ such that*

$$\lim_{n\to\infty} \sup_{\tau \in QT(A)} |\tau(h_n \circ L_n(a)) - \tau(a)| = 0 \tag{e 6.248}$$

for all $a \in A$, and for each projection $p \in A$, there exists a sequence of projection $p_n \in A_n$ such that

$$\lim_{n\to\infty} \|L_n(p) - p_n\| = 0. \tag{e 6.249}$$

Note that the existence of projections $p_n \in A_n$ can be easily constructed from the construction in the proof of 6.1.

Corollary 6.3. *Let A be a unital simple C^*-algebra in $\mathcal{C}_1$. Then every quasitrace extends a trace.*

Proof. Let $\tau : A_+ \to \mathbb{R}_+$ be a quasitrace in $QT(A)$. It is known that every quasitrace on A_n extends a trace. We will use the notation in the proof of 6.1. Thus $\tau \circ h_n$ is a trace. If $x, y \in A_+$, then

$$\begin{aligned} \tau \circ h_n(L_n(x+y)) &= \tau \circ h_n(L_n(x) + L_n(y)) & \text{(e 6.250)} \\ &= \tau \circ h_n(L_n(x)) + \tau \circ h_n(L_n(y)). & \text{(e 6.251)} \end{aligned}$$

So $\tau \circ h_n \circ L_n$ extends a state. Let t be a weak limit of $\{\tau \circ L_n \varphi_n\}$. By (e 6.247), $t(x) = \tau(x)$ for all $x \in A_+$. ☐

In the following, if Ω is a compact convex set, $\partial_e(\Omega)$ is the set of extremal points of Ω.

Theorem 6.4. *Let A be a unital separable simple C^*-algebra in $\mathcal{C}_1$. Then $r_A(\partial_e(T(A)) = \partial_e(S(K_0(A)))$.*

Proof. Note that, by 6.3, $T(A) = QT(A)$. It follows from 6.1 of [50] that r_A is surjective and $\partial_e(S(K_0(A))) \subset r_A(\partial_e(T(A)))$. We will prove that $r_A(\partial_e(T(A))) \subset \partial_e(S(K_0(A)))$.

Suppose $\tau \in \partial_e(T(A))$ and there are $s_1, s_2 \in S(K_0(A))$ such that

$$r_A(\tau) = ts_1 + (1-t)s_2$$

for some $t \in (0,1)$. Suppose that $s_1 \neq s_2$. Then, since $A \in \mathcal{C}_1$, there is a projection $p \in A$ such that

$$s_1([p]) \neq s_2([p]). \tag{e 6.252}$$

Let $A_n \in \mathcal{I}^{k(n)}$, L_n and h_n be as in 6.2. In particular, there are projections $p_n \in A_n$ such that

$$\lim_{n\to\infty} \|L_n(p) - p_n\| = 0 \text{ and} \tag{e 6.253}$$

$$\lim_{n\to\infty} |\tau'(h_n(p_n)) - \tau'(p)| = 0 \text{ for all } \tau' \in T(A). \tag{e 6.254}$$

Moreover,

$$\lim_{n\to\infty} |\tau'(h_n(1_{A_n})) - 1| = 0 \text{ for all } \tau' \in T(A). \tag{e 6.255}$$

For each n,

$$\tau(h_n(q)) = ts_1([h_n(q)]) + (1-t)s_2([h_n(q)]) \tag{e 6.256}$$

for all projections $q \in A_n \otimes \mathcal{K}$. Write $A_n = C_{n,1} \oplus C_{n,2} \oplus \cdots \oplus C_{n,k(n)}$, where each $C_{n,i} = P_{n,i} M_{r(n,i)}(C(X_{n,i}))P_{n,i}$ and $X_{n,i}$ is connected. Note that $\rho_{C_{n,i}}(K_0(C_{n,i})) = \mathbb{Z}$. We may assume that $h_n(C_{n,i}) \neq 0$ (otherwise, we delete that summand). Therefore there are $0 \le a_{n,i}, \beta_{n,i}$ such that

$$\begin{aligned} a_{n,i}\tau \circ h_n|_{K_0(C_{n,i})} &= (s_1 \circ [h_n])|_{K_0(C_{n,i})} \text{ and} & \text{(e 6.257)} \\ b_{n,i}\tau \circ h_n|_{K_0(C_{n,i})} &= (s_2 \circ [h_n])|_{K_0(C_{n,i})}, & \text{(e 6.258)} \end{aligned}$$

$i = 1, 2, ..., k(n)$ and $n = 1, 2, ...$ Since $r_A(\tau) = ts_1 + (1-t)s_2$,

$$ta_{n,i}\tau \circ h_n(1_{C_{n,i}}) + (1-t)b_{n,i}\tau \circ h_n(1_{C_{n,i}}) \tag{e 6.259}$$
$$= ts_1 \circ [h_n(1_{C_{n,i}}]) + (1-t)s_2 \circ [h_n(1_{C_{n,i}}]) \tag{e 6.260}$$
$$= \tau(h_n(1_{C_{n,i}})). \tag{e 6.261}$$

It follows that

$$ta_{n,i} + (1-t)b_{n,i} = 1. \tag{e 6.262}$$

Note that

$$\sum_{i=1}^{k(n)} a_{n,i}\tau(h_n(1_{C_{n,i}})) = s_1([h_n(1_{A_n})]) \text{ and } \sum_{i=1}^{k(n)} b_{n,i}\tau(h_n(1_{C_{n,i}})) = s_2([h_n(1_{A_n})]). \tag{e 6.263}$$

Put

$$a_n = \sum_{i=1}^{k(n)} a_{n,i}\tau(h_n(1_{C_{n,i}})) \text{ and } b_n = \sum_{i=1}^{k(n)} b_{n,i}\tau(h_n(1_{C_{n,i}})).$$

By (e 6.255),

$$\lim_{n\to\infty} a_n = 1 \text{ and } \lim_{n\to\infty} b_n = 1. \tag{e 6.264}$$

Since s_1 and s_2 are states on $K_0(A)$ and A is simple, $a_n > 0$ and $b_n > 0$. Let $\pi_{n,i} : A_n \to C_{n,i}$ be the projection map. Define

$$\tau^{(n,1)} = (\frac{1}{a_n})\sum_{i=1}^{k(n)} a_{n,i}\tau \circ h_n|_{C_{n,i}} \circ \pi_{n,i} \text{ and} \tag{e 6.265}$$
$$\tau^{(n,2)} = (\frac{1}{b_n})\sum_{i=1}^{k(n)} b_{n,i}\tau \circ h_n|_{C_{n,i}} \circ \pi_{n,i}. \tag{e 6.266}$$

Therefore $\tau^{(n,1)}$ and $\tau^{(n,2)}$ are tracial states on A_n. By (e 6.262),

$$\tau|_{h_n(A_n)} = t(\sum_{i=1}^{k(n)} a_{n,i}\tau \circ h_n|_{C_{n,i}} \circ \pi_{n,i}) + (1-t)(\sum_{i=1}^{k(n)} b_{n,i}\tau \circ h_n|_{C_{n,i}} \circ \pi_{n,i}) \tag{e 6.267}$$

By the definition of L_n, (e 6.254), (e 6.264), (e 6.265) and (e 6.266),

$$\tau(a) = \lim_{n\to\infty} \tau(h_n \circ L_n(a)) \tag{e 6.268}$$
$$= \lim_{n\to\infty} [t\tau^{(n,1)}(L_n(a)) + (1-t)\tau^{(n,2)}(L_n(a))] \tag{e 6.269}$$

for all $a \in A$. Note that $\tau^{(n,i)} \circ L_n$ is a state on A, $i = 1, 2$. Let τ_1 and τ_2 be limit points of $\{\tau^{(n,1)} \circ L_n\}$ and $\{\tau^{(n,2)} \circ L_n\}$, respectively. One checks easily that both are tracial states of A. By (e 6.269),

$$\tau = t\tau_1 + (1-t)\tau_2. \tag{e 6.270}$$

Since $\tau \in \partial_e(T(A))$, this implies that

$$\tau = \tau_1 = \tau_2. \tag{e 6.271}$$

On the other hand,

$$\tau_1(p) = \lim_{n\to\infty} \tau^{(n,i)}(L_n(p)) \quad = \quad \lim_{n\to\infty} s_1([p_n]) = s_1([p]) \text{ and} \tag{e 6.272}$$

$$\tau_2(p) = \lim_{n\to\infty} \tau^{(n,2)}(L_n(p)) \quad = \quad \lim_{n\to\infty} s_2([p_n]) = s_2([p]). \tag{e 6.273}$$

This contradicts the assumption that $s_1([p]) \neq s_2([p])$. It follows that $r_A(\tau) \in \partial_e(K_0(A))$. □

The following is a variation of Theorem 5.3 of [6].

Theorem 6.5. *Let A be a unital separable simple C^*-algebra in $\mathcal{C}_1$. Then the map $W(A) = V(A) \sqcup \mathrm{LAff}_b(T(A))$.*

Proof. Note that $QT(A) = T(A)$. By Theorem 4.4 of [41] (see also Theorem 2.2 of [6]), it suffices to prove that the map from $W(A)$ to $V(A) \sqcup \mathrm{LAff}_b(T(A))$ is surjective. The proof of that is a slight modification of that of Theorem 5.3 of [6]. We also apply Lemma 5.2 in [6]. The only difference is that, in the proof of Theorem 5.3 of [6], at the point 5.1 of [6] is used in the proof, we use 6.1 instead. □

Corollary 6.6. *Let A be a unital separable simple C^*-algebra in $\mathcal{C}_1$. Then A has 0-almost divisible Cuntz semigroup (see definition 2.5 of [60]).*

Proof. Let $a \in M_\infty(A)$ and $k \geq 1$ be an integer. If $\langle a \rangle$ is represented by a projection $p \in M_m(A)$, then by 5.7, since $M_m(A)$ is in $\mathcal{C}_1$, there is a projection $p_1 \in M_m(A)$ such that

$$k[p_1] \leq [p] \leq (k+1)[p_1].$$

Now suppose that a can not be represented by a projection. Then, 6.5, $\langle a \rangle \in \mathrm{LAff}_b(T(A))$. Note that the function $\langle a \rangle/k \in \mathrm{LAff}_b(T(A))$. It follows from 6.5 that there is $x \in W(A)$ such that $x = \langle a \rangle/k$. Then

$$kx \leq \langle a \rangle \leq (k+1)x.$$

□

Lemma 6.7. *Let $C = \lim_{n\to\infty}(C_n, \psi_n)$ be a unital C^*-algebra such that each C_n is a separable unital amenable C^*-algebra with $T(C_n) \neq \emptyset$ and each ψ_n is unital homomorphism. Let X be a compact metric space and $\Delta : (0,1) \to (0,1)$ be a non-decreasing map. Suppose that $\varphi : C(X) \to C$ is a contractive completely positive linear map and*

$$\mu_{\tau\circ\varphi}(O_r) \geq \Delta(r) \text{ for all } \tau \in T(C) \tag{e 6.274}$$

for all open balls O_r with radius $r \geq \eta$ for some $\eta \in (0, 1/9)$.

Then, for any $\epsilon > 0$, any finite subset $\mathcal{F} \subset C(X)$, any finite subset $\mathcal{G} \subset C$, there exists an integer $n \geq 1$, a unital contractive completely positive linear map $L : C(X) \to C_n$ and a unital contractive completely positive linear map $r : C \to C_n$ such that $L = r \circ \varphi$,

$$\|\psi_{n,\infty} \circ L(f) - \varphi(f)\| \quad < \quad \epsilon \text{ for all } f \in \mathcal{F}, \tag{e 6.275}$$

$$\|\psi_{n,\infty} \circ r(g) - g\| \quad < \quad \epsilon \text{ for all } f \in \mathcal{G} \text{ and} \tag{e 6.276}$$

$$\mu_{t\circ L}(O_r) \quad \geq \quad \Delta(r/3)/3 \text{ for all } t \in T(C_n) \tag{e 6.277}$$

for all $r \geq 17\eta/8$. Furthermore, if φ is $\epsilon/2$-$\mathcal{F}$-multiplicative, we may also require that L is ϵ-$\mathcal{F}$-multiplicative.

Proof. Since each C_n is amenable, by applying 2.3.13 of [23], there exists a sequence of contractive completely positive linear map $r_n : C \to C_n$ such that

$$\lim_{n\to\infty} \|\psi_{n,\infty} \circ r_n(x) - x\| = 0 \text{ for all } x \in C. \tag{e 6.278}$$

Define $L_n = r_n \circ \varphi$, $n = 1, 2, ...$ Using the definition of inductive limits, for any sufficiently large n, L_n can be chosen as L and r_n can be chosen as r to satisfy (e 6.275) and (e 6.276) as well as the requirement that L is ϵ-$\mathcal{F}$-multiplicative, provided that φ is $\epsilon/2$-$\mathcal{F}$-multiplicative. To see that we can also find L so (e 6.277) holds, we will prove the following: for any integer $k \geq 1$, there is an integer $n \geq k$ such that, if $f \in C(X)$ with $0 \leq f \leq 1$ and $\{x : f(x) = 1\}$ contains an open ball O_s with radius $s \geq 17\eta/8$, then

$$t \circ L_n(f) \geq \Delta(s/3)/3 \text{ for all } t \in T(C_n). \tag{e 6.279}$$

This will imply (e 6.277).

Otherwise, there is a sequence $\{k(n)\}$ with $\lim_{n\to\infty} k(n) = \infty$, there is $t_n \in T(C_{k(n)})$ and $s_n \in (17\eta/8, 1/2)$ such that

$$t_n \circ L_{k(n)}(f_n) < \Delta(s/3)/3 \tag{e 6.280}$$

for all n and for some $f_n \in C(X)$ with $0 \leq f \leq 1$ and $\{x : f_n(x) = 1\}$ contains an open ball with radius $s_n \geq 17\eta/8$. Note that $\{t_n \circ r_{k(n)}\}$ is a sequence of states of C. Let t_0 be a weak limit point of $\{t_n \circ r_{k(n)}\}$. We may assume, that $t_0(a) = \lim_{m\to\infty} t_{n(m)} \circ r_{k(n(m))}(a)$ for all $a \in C$. By passing to a subsequence, we may assume that $s_{n(m)} \to s$, where $s \in [17\eta/8, 1/2]$.

Let $x_1, x_2, ..., x_l$ be a set of finite points of X such that $\cup_{i=1}^{l} O(x_i, 15s/32) \supset X$, where $O(x_i, 15s/32)$ is the open ball with center x_i and radius $15s/32$. We may assume that $17s/16 > s_{n(m)} > 15s/16 + s/2^8$. Note that $30s/32 \geq \eta$. Denote by $f_i \in C(X)$ with $0 \leq f \leq 1$ and $f_l(x) = 1$ if $x \in O(x_i, 15s/32)$ and $f_l(x) = 0$ if $\mathrm{dist}(x, x_i) \geq 15s/32 + s/2^8$, $i = 1, 2, ..., l$. It follows that there are infinitely many $f_{n(m)}$ such that $f_{n(m)} \geq f_j$ for some $j \in \{1, 2, ..., l\}$. To simplify notation, by passing to a subsequence, we may assume, for all m, $f_{n(m)} \geq f_j$. By (e 6.280),

$$t_{n(m)} \circ L_{k(n(m))}(f_j) < \Delta(s/3)/3. \tag{e 6.281}$$

One verifies (since $C = \lim_{n\to\infty}(C_n, \psi_n)$), by (e 6.276), t_0 is a tracial state. It follows from (e 6.274) that

$$t_0(f) \geq \Delta(15s/32) \tag{e 6.282}$$

for all $f \in C(X)$ with $0 \leq f \leq 1$ and $\{x : f(x) = 1\}$ contains an open ball $O_{15s/32}$ with radius $15s/32$. However, by (e 6.281),

$$t_0(f_j) < \Delta(s/3)/3. \tag{e 6.283}$$

A contradiction.

□

6.8. *Let $C(X)$ be a compact metric space. Suppose that $\varphi : C(X) \to A$ is a monomorphism. Then there is an nondecreasing map $\Delta : (0, 1) \to (0, 1)$ such that*

$$\mu_{\tau\circ\varphi}(O_r) \geq \Delta(r) \tag{e 6.284}$$

for all $\tau \in T(A)$ and all open balls with radius $r > 0$ (see 6.1 of [36]).

The following statement can be easily proved by the argument used in the proof of 6.7 and that of (e 6.232) of 6.1.

Lemma 6.9. *Let $\mathcal{B}$ be a class of unital separable amenable C^*-algebra B with $T(B) \neq \emptyset$. Let A be a unital separable simple C^*-algebra which is tracially $\mathcal{B}$. Let X be a compact metric space, Suppose that $\varphi : C(X) \to A$ is a unital monomorphism with*

$$\mu_{\tau\circ\varphi}(O_r) \geq \Delta(r) \text{ for all } \tau \in T(A) \tag{e 6.285}$$

and for all open balls with radius $r > 0$. Then, for any $a \in A_+ \setminus \{0\}$, any $\eta > 0$, $\delta > 0$ and any finite subset $\mathcal{G} \subset C(X)$, there exists a projection $p \in A$, a unital C^*-subalgebra $B \in \mathcal{B}$ with $1_B = p$ and a unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map $\Phi : C(X) \to B$ such that

$$\|p\varphi(x) - \varphi(x)p\| < \delta \text{ for all } x \in \mathcal{G}, \tag{e 6.286}$$

$$\|p\varphi(x)p - \Phi(x)\| < \delta \text{ for all } x \in \mathcal{G} \text{ and} \tag{e 6.287}$$

$$\mu_{\tau\circ\Phi}(O_r) \geq \Delta(r/3)/3 \text{ for all } \tau \in T(B) \tag{e 6.288}$$

and for all opens balls O_r with radius $r \geq \eta$.

7 The unitary group

Definition 7.1. For each integer $d \geq 1$, let $K(d)$ be an integer associated with d given by Lemma 3.4 of [42]. It should be noted (from the proof of Lemma 3.4 of [42]), or by choosing even large $K(d)$, applying [45]) that, if $K \geq K(d)$, then, for any compact metric space X with covering dimension d, any projection $p \in M_N(C(X))$ (for some integer $N \geq K$) with rank p at least K at each point x and any unitary $u \in U_0(pM_NC(X))p)$, there are selfadjoint elements $h_1, h_2, h_3 \in (pM_N(C(X))p)$ such that

$$\|u - \exp(ih_1)\exp(ih_2)\exp(ih_3)\| < 1.$$

Let $g(d) = K(d)$ for all $d \in \mathbb{N}$. Put $\mathcal{C}_{1,1} = \mathcal{C}_g$.

Proposition 7.2. *Let A be a unital simple C^*-algebra in $\mathcal{C}_{1,1}$ and let $u \in U_0(A)$. Then, for any $\epsilon > 0$, there are four unitaries $u_0, u_1, u_2, u_3 \in A$, such that u_1, u_2, u_3 are exponentials and u_0 is a unitary with $\mathrm{cel}(u_0) \leq 2\pi$ such that*

$$\|u - u_0u_1u_2u_3u_4\| < \epsilon/2. \tag{e 7.289}$$

Moreover, $\mathrm{cer}(A) \leq 6 + \epsilon$.

Proof. Let $u \in U_0(A)$. Then, for any $\pi/4 > \epsilon > 0$, there are unitaries $v_1, v_2, ..., v_n \in U(A)$ such that

$$v_1 = u, \quad v_n = 1 \text{ and } \|v_{i+1} - v_i\| < \epsilon/16, \quad i = 1, 2, ..., n. \tag{e 7.290}$$

Since A is simple and has (SP), there are mutually orthogonal and mutually equivalent non-zero projections $q_1, q_2, ..., q_{4(n+1)} \in A$. For each integer $d \geq 1$, let $K(d)$ be an integer given by 7.1. Since $A \in \mathcal{C}_{1,1}$, there exists a projection $p \in A$ and a C^*-subalgebra $C = PM_r(C(X))P$, where $r \ge 1$ is an integer, X is a compact subset of a $d(X)$ dimensional CW complex and $P \in M_r(C(X))$ is a projection such that

$$\|pv_i - v_ip\| < \epsilon/128, \quad i = 1, 2, ..., n \tag{e 7.291}$$
$$\mathrm{dist}(pv_ip, C) < \epsilon/128, \quad i = 1, 2, ..., n \tag{e 7.292}$$
$$\frac{d(X)+1}{\mathrm{rank}(P(x))} < \frac{d}{64K(d)} \text{ for all } x \in X \text{ and} \tag{e 7.293}$$
$$1 - p \lesssim q_1. \tag{e 7.294}$$

There are unitaries $w_i \in (1-p)A(1-p)$ with $w_0 = (1-p)$ such that

$$\|w_i - (1-p)v_i(1-p)\| < \epsilon/16, \quad i = 1, 2, ..., n. \tag{e 7.295}$$

Furthermore, there is a unitary $z \in C$ such that

$$\|z - pup\| < \epsilon/16 \tag{e 7.296}$$

Therefore

$$\|u - (w_1 \oplus z)\| < \epsilon/8. \tag{e 7.297}$$

Put $z_1 = w_1 + p$. Since $1 - p \lesssim q_1$, by Lemma 6.4 of [29], $\mathrm{cel}(z_1) \le 2\pi + \epsilon/4$. By the choice of $K(d)$ (see 7.1), there are three self-adjoint elements $h_1, h_2, h_3 \in B$ such that

$$\|z - \exp(ih_1)\exp(ih_2)\exp(ih_3)\| < 1. \tag{e 7.298}$$

Let $u_j = \exp(ih_j) + (1-p)$, $j = 1, 2, 3$. Then,

$$\|u - z_1u_1u_2u_3\| < 1 + \epsilon/4 \tag{e 7.299}$$

Moreover, since $\mathrm{cel}(z_1) \le 2\pi + \epsilon/4$, there is a unitary u_0 such that $\mathrm{cel}(u_0) \le 2\pi$ and

$$\|u_0 - z_1\| < \epsilon/2.$$

It follows that $\mathrm{cer}(A) \le 6 + \epsilon$.

□

Theorem 7.3. *Let A be a unital simple C^*-algebra in $\mathcal{C}_{1,1}$. Suppose that $u \in CU(A)$. Then, $u \in U_0(A)$ and*

$$\mathrm{cel}(u) \le 8\pi. \tag{e 7.300}$$

Proof. Since $u \in CU(A)$, it is easy to show that $[u] = 0$ in $K_1(A)$. It follows from 5.2 that A has stable rank one. Therefore $u \in U_0(A)$. Now let $\Lambda = \mathrm{cel}(u)$. Let $\epsilon > 0$. Choose an integer $K_1 \ge 1$ such that $(\Lambda + 1)/K_1 < \epsilon/4$. There exists a unitary $v \in U_0(A)$ which is a finite product of commutators such that

$$\|u - v\| < \epsilon/16. \tag{e 7.301}$$

Fix a finite subset $\mathcal{F} \subset A$ which contains u and v and among other elements. Let $\delta > 0$. Since A is in $\mathcal{C}_{1,1}$, there is a projection $p \in A$ and C^*-subalgebra $C \subset A$ with $1_C = p$ and $C = PM_r(X)P$, where X is a compact subset of a finite CW complex with dimension d, $r \ge 1$ is an integer, $P \in M_r(X)$ is a projection with

$$\mathrm{rank}P(\xi) > K(d) \text{ for all } \xi \in X, \tag{e 7.302}$$

where $K(d)$ is the integer given by Lemma 3.4 of [42], such that

$$\|px - xp\| < \delta \text{ for all } x \in \mathcal{F}, \tag{e 7.303}$$
$$\mathrm{dist}(pxp, C) < \epsilon/16 \text{ for all } x \in \mathcal{F} \text{ and} \tag{e 7.304}$$

$$(K_1 + 1)[1 - p] \le [p]$$

By choosing sufficiently large $\mathcal{F}$ and sufficiently small δ, we obtain unitaries $u_1 \in (1-p)A(1-p)$ and a unitary $v_1 \in U_0(C)$ which is a finite product of commutators of unitaries in C such that

$$\|u - (u_1 + v_1)\| < \epsilon/8 \text{ and} \tag{e 7.305}$$
$$\mathrm{cel}(u_1) \le \Lambda + 1 \text{ (in } (1-p)A(1-p)). \tag{e 7.306}$$

Write $w = u_1 + p$. It follows from 6.4 of [29] that

$$\mathrm{cel}(w) < 2\pi + \epsilon/4. \tag{e 7.307}$$

Since $v_1 \in U_0(A)$ is a finite product of commutators of unitaries in C, by the choice of $K(d)$ and by applying 3.4 of [42],

$$\mathrm{cel}(v_1) \le 6\pi + \epsilon/8 \tag{e 7.308}$$

(in C). Thus

$$\mathrm{cel}(v_1 + (1-p)) \le 6\pi + \epsilon/4. \tag{e 7.309}$$

It follows from (e 7.305), (e 7.307) and (e 7.309) that

$$\mathrm{cel}(u_1 + v_1) \le 2\pi + \epsilon/4 + 6\pi + \epsilon/4. \tag{e 7.310}$$

By (e 7.305),

$$\mathrm{cel}(u) \le (\epsilon/16)\pi + 8\pi + \epsilon/2 \le 8\pi + \epsilon.$$

□

Theorem 7.4. *Let A be a unital simple C^*-algebra in $\mathcal{C}_{1,1}$ and let $k \ge 1$ be an integer. Suppose that $u, v \in U(A)$ such that $[u] = [v]$ in $K_1(A)$,*

$$u^k, \ v^k \in U_0(A) \text{ and } \mathrm{cel}((u^k)^* v^k) \le L \tag{e 7.311}$$

for some $L > 0$. Then

$$\mathrm{cel}(u^* v) \le L/k + 8\pi. \tag{e 7.312}$$

Proof. Suppose that

$$u^* v = \prod_{j=1}^{R(u)} \exp(i a_j) \text{ and } (u^*)^k v^k = \prod_{m}^{R(v)} \exp(i b_m),$$

where $a_j, b_m \in A_{s.a.}$. Since $\mathrm{cel}((u^*)^k v^k) \le L$, we may assume that $\sum_m \|b_m\| \le L$ (see [48]). Let $M = \sum_j \|a_j\|$. In particular, $\mathrm{cel}(u^* v) \le M$.

Let $\epsilon > 0$. Let $\delta > 0$ be such that $\frac{\delta}{1-\delta} < \frac{\epsilon}{2(M+L+1)}$. Let $\eta > 0$. Since $A \in \mathcal{C}_{1,1}$, there exists a projection $p \in A$ and a C^*-subalgebra $C \in \mathcal{I}_d$, where $C = PM_r(C(X))P$ and $1_C = p$ such that

$$\|u - u_0 \oplus u_1\| < \eta \text{ and } \|v - v_0 \oplus v_1\| < \eta, \tag{e 7.313}$$
$$u_0, v_0 \in U((1-p)A(1-p)), \;\; u_1, v_1 \in U(C), \tag{e 7.314}$$
$$u_0^* v_0 = \prod_{j=1}^{R(u)} \exp(ipa_j p), \;\; (u_1^*)^k v_1^k = \prod_{m=1}^{R(v)} \exp(ib'_m), \tag{e 7.315}$$
$$\frac{d+1}{\mathrm{rank}P(x)} \le \min\{2\pi^2/\epsilon, d/K(d)+1\} \text{ for all } x \in X; \tag{e 7.316}$$
$$\tau(1-p) < \delta \text{ for all } \tau \in T(A), \tag{e 7.317}$$

where $b'_m \in C_{s.a.}$ and $\|b'_m\| \le L + \epsilon/4$.

Note that $\mathrm{rank}P$ is a continuous function on X.

It follows from 3.3 (1) of [42] that there exists $a \in C_{s.a.}$ with $\|a\| \le L + \epsilon/4$ such that

$$\det(\exp(ia)(u_1^*)^k v_1^k) = 1 \text{ (for all } x \in X). \tag{e 7.318}$$

Therefore

$$\det((\exp(ia/k)u_1^* v_1)^k) = 1 \text{ for all } x \in X. \tag{e 7.319}$$

It follows that

$$\det(\exp(ia/k)u_1^* v_1)(x) = \exp(i2l(x)\pi/k) \text{ for all } x \in X \tag{e 7.320}$$

for some continuous function $l(x)$ on X with $l(x)$ being an integer. Since X is compact, $l(x)$ has only finitely many values. We may choose these values among $0, 1, ..., k-1$. Let $f(x) = -2l(x)\pi/k$ for $x \in X$. Then $f \in C(X)_{s.a.}$ and $\|f\| \le 2\pi$. Note that $\exp(if/\mathrm{rank}P) \cdot 1_C$ commutes with $\exp(ia/k)$ and

$$\exp(i(f/\mathrm{rank}P(x)) \cdot 1_C) \exp(ia/k) = \exp((i((f/\mathrm{rank}P(x)) + a/k)). \tag{e 7.321}$$

We have that

$$\det(\exp(i(f/\mathrm{rank}P(x) + a/k))u_1^* v_1) = 1. \tag{e 7.322}$$

It follows from 3.4 of [42] that

$$\mathrm{cel}(u_1^* v_1) \le 2\pi/(2\pi^2/\epsilon) + (L + \epsilon/4)/k + 6\pi. \tag{e 7.323}$$

By applying Lemma 6.4 of [29],

$$\mathrm{cel}((u_0 \oplus p)^*(v_0 \oplus p)) \le 2\pi + \epsilon/2. \tag{e 7.324}$$

It follows that, with sufficiently small η,

$$\mathrm{cel}(u^* v) \le 2\pi + \epsilon/2 + \epsilon/\pi + L/k + \epsilon/4k + 6\pi. \tag{e 7.325}$$

The lemma follows.

□

Theorem 7.5. *Let A be a unital infinite dimensional simple C^*-algebra in $\mathcal{C}_1$. Then $U_0(A)/CU(A)$ is a torsion free and divisible group.*

Proof. We have shown that A has stable rank one. Therefore, by Thomsen's result ([52]), $U_0(A)/CU(A) \cong \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$. In particular, $U_0(A)/CU(A)$ is divisible. To show that it is torsion free, we assume that $x \in \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$ such that $kx = 0$ for some integer $k \geq 1$. Let $y \in \mathrm{Aff}(T(A))$ such that $\bar{y} = x$ in the quotient. So $ky \in \overline{\rho_A(K_0(A))}$.

Let $\epsilon > 0$. There is an element $z \in K_0(A)$ such that

$$\sup_{\tau \in T(A)} |ky(\tau) - \rho_A(z)(\tau)| < \epsilon/2. \tag{e 7.326}$$

We may assume that $z = [p] - [q]$, where $p, q \in M_n(A)$ are two projections for some integer $n \geq 1$. By (SP), there is a non-zero projection $e \in A$ such that $\tau(e) < \epsilon/4k$ for all $\tau \in T(A)$. It follows from 5.7 that there are mutually orthogonal projections $p_1, p_2, ..., p_{k+1} \in pM_n(A)p$ and mutually orthogonal projections $q_1, q_2, ..., q_{k+1} \in qM_n(A)q$ such that

$$[p_{k+1}] \leq [e], \;\; [p_1] = [p_i], \;\; i = 1, 2, ..., k, \;\; \text{and} \tag{e 7.327}$$

$$[q_{k+1}] \leq [e], \;\; [q_1] = [q_i], \;\; i = 1, 2, ..., k. \tag{e 7.328}$$

Then, by (e 7.326),

$$\sup_{\tau \in T(A)} |y(\tau) - \rho_A([p_1] - [q_1])(\tau)| < \epsilon/2k + \epsilon/4k < \epsilon. \tag{e 7.329}$$

It follows that $y \in \overline{\rho_A(K_0(A))}$. This implies that $x = 0$. Therefore that $U_0(A)/CU(A)$ is torsion free.

□

Theorem 7.6. *Let A be a unital simple C^*-algebra with stable rank one and let $e \in A$ be a non-zero projection. Then the map $u \mapsto u + (1 - e)$ induces an isomorphism from $U(eAe)/CU(eAe)$ onto $U(A)/CU(A)$.*

Proof. Note, by the assumption that A has stable rank one, $CU(eAe) \subset U_0(eAe)$ and $CU(A) \subset U_0(A)$. We define a map $\mathrm{Aff}(T(eAe))$ to $\mathrm{Aff}(T(A))$ as follows. Let $\Lambda_B : B_{s.a} \to \mathrm{Aff}(T(B))$ by $\Lambda_B(b)(\tau) = b(\tau)$ for all $\tau \in T(B)$ and $b \in B_{s.a.}$, where $b \in B$ and B is a C^*-algebra. By [8], we will identify $\mathrm{Aff}(T(A))$ with $\Lambda_A(A_{s.a.})$ and $\mathrm{Aff}(T(eAe))$ with $\Lambda_{eAe}((eAe)_{s.a.})$.

Define $\gamma : \mathrm{Aff}(T(eAe)) \to \mathrm{Aff}(A)$ by $\gamma(a)(\tau) = \tau(a)$ for all $\tau \in T(A)$ and $a \in (eAe)_{s.a.}$. Clearly γ is homomorphism. Since A is simple, γ maps $\rho_{eAe}(K_0(A))$ into $\rho_A(K_0(A))$. Hence it induces a homomorphism

$$\bar{\gamma} : \mathrm{Aff}(T(eAe))/\overline{\rho_{eAe}(K_0(A))} \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}.$$

Since A is simple, $\bar{\gamma}$ is injective.

To see it is also surjective, let $h \in A_{s.a.}$. Since A is simple, there is an integer $K \geq 1$ such that

$$N[1_A] \geq K[e] \geq [1_A] \tag{e 7.330}$$

for some integer N. Then, there is a partial isometry $U \in M_N(A)$ such that

$$U^*U = 1_A \;\; \text{and} \;\; UU^* \leq \mathrm{diag}(\overbrace{e, e, ..., e}^{K}). \tag{e 7.331}$$

Let $E = \mathrm{diag}(\overbrace{e, e, ..., e}^{K})$. We will identify $EM_N(A)E$ with $M_K(eAe)$. Write $UhU^* = (h_{i,j})$, a $K \times K$ matrix in $M_K(eAe)$. Write $h = h_+ - h_-$ and $h_+ = (h_{i,j}^+)$ and $h_- = (h_{i,j}^-)$.

It is known (see 2.2.2 of [23], for example) that there are $h_{+,k}=(h^+_{k,i,j})$ and $h_{-,k}=(h^-_{k,i,j})$, $k=1,2,...,n$ and $x_1,x_2,...,x_n,y_1,y_2,...,y_n\in M_K(eAe)$ (for some integer $n\ge 1$) such that

$$h_+=\sum_{k=1}^n h_{+,k},\ \ h_-=\sum_{k=1}^n h_{-,k}, \tag{e 7.332}$$

$$x_k^*x_k=h_{+,k},\ \ x_kx_k^*=\sum_{j=1}^n h^+_{k,j,j} \tag{e 7.333}$$

$$y_k^*y_k=h_{-,k}\ \text{ and }\ y_ky_k^*=\sum_{j=1}^n h^-_{k,j,j}. \tag{e 7.334}$$

Note $b=\sum_{k=1}^n x_kx_k^*-\sum_{k=1}^n y_ky_k^*\in eAe$. We have

$$\tau(b)=\tau(h_+-h_-)=\tau(h)\ \text{ for all }\ \tau\in T(A). \tag{e 7.335}$$

It follows that $\overline{\gamma}$ is surjective. From this and by a theorem of Thomsen ([52]), the map $u\mapsto u+(1-e)$ is an isomorphism from $U_0(eAe)/CU(eAe)$ onto $U_0(A)/CU(A)$. Since A has stable rank one, $U(eAe)/U_0(eAe)=U(A)/U_0(A)$. It follows that $u\to u+(1-e)$ is an isomorphism from $U(eAe)/CU(eAe)$ onto $U(A)/U_0(A)$. □

Lemma 7.7. *Let $K>1$ be an integer. Let A be a unital simple C^*-algebra, let $e\in A$ be a projection, let $u\in U_0(eAe)$ and let $w=u+(1-e)$. Suppose that $\eta>0$,*

$$\mathrm{dist}(\bar{w},\bar{1})<\eta \tag{e 7.336}$$

and $1-e\lesssim \mathrm{diag}(\overbrace{e,e,...,e}^{K-1})$. *Then, if $\eta<2$,*

$$\mathrm{dist}_{eAe}(\bar{u},\bar{e})<K\eta; \tag{e 7.337}$$

If furthermore, $A\in\mathcal{C}_{1,1}$, then

$$\mathrm{cel}_{eAe}(u)\le K\eta+8\pi. \tag{e 7.338}$$

If $\eta=2$ and $A\in\mathcal{C}_{1,1}$, then

$$\mathrm{cel}_{eAe}(u)\le K\mathrm{cel}(w)+8\pi. \tag{e 7.339}$$

Proof. We first consider the case that $\eta<2$. Let $\epsilon>0$ such that $\eta+\epsilon<2$. There is $w_1\in U(A)$ such that $\overline{w_1}=\overline{w}$ and

$$\|w_1-1\|<\eta+\epsilon/2<2. \tag{e 7.340}$$

Thus there is $h\in A_{s.a}$ such that

$$w_1=\exp(ih)\ \text{ and }\ \|h\|<2\arcsin(\frac{\eta+\epsilon/2}{2})<\pi. \tag{e 7.341}$$

It follows that

$$\|w_1-1\|=\|\exp(ih)-1\|=|\exp(i\|h\|)-1|. \tag{e 7.342}$$

Since

$$1-e\lesssim\mathrm{diag}(\overbrace{e,e,...,e}^{K-1}),$$

we may view h as an element in $M_K(eAe)$ (see the proof of 7.4). It follows from 7.6 that there is $f \in \mathrm{Aff}(T(eAe))$ such that $f(\tau) = \tau(h)$ for all $\tau \in T(eAe)$. Note that $\tau(h) = (\tau \otimes Tr_K)(h)$ for $\tau \in T(eAe)$. Since A is simple, by [8] (see also 9.3 of [29]), there exists $a \in (eAe)_{s.a.}$ such that

$$\tau(a) = f(\tau) \text{ for all } \tau \in T(eAe) \text{ and } \|a\| < \|f\| + \delta \qquad \text{(e 7.343)}$$

for some $\delta > 0$ such that

$$2K \arcsin(\frac{\eta + \epsilon/2}{2}) + \delta < 2K \arcsin(\frac{\eta + \epsilon}{2}). \qquad \text{(e 7.344)}$$

It follows that

$$\overline{\Delta}_A(\exp(ia)(u^* + (1 - e)) = 0. \qquad \text{(e 7.345)}$$

Note that

$$(1 - e)(\exp(ia)u^*) = (\exp(ia)u^*)(1 - e) = 1 - e \text{ and} \qquad \text{(e 7.346)}$$

$$e \exp(ia) = \exp(ia)e = e + \sum_{i=1}^{\infty} \frac{(ia)^k}{k!}. \qquad \text{(e 7.347)}$$

Therefore

$$\overline{\Delta}_{eAe}(e \exp(ia)u^*) = 0. \qquad \text{(e 7.348)}$$

Thus, by [52],

$$\overline{e \exp(ia)} = \overline{u} \text{ in } U(eAe)/CU(eAe). \qquad \text{(e 7.349)}$$

Note that

$$\|a\| \le \|f\| + \delta \le \sup\{\frac{|\tau(h)|}{\tau(e)} : \tau \in T(A)\} + \epsilon/2 \qquad \text{(e 7.350)}$$

$$\le K\|h\| + \delta \le K(2 \arcsin(\frac{\eta + \epsilon}{2})). \qquad \text{(e 7.351)}$$

If

$$K(2 \arcsin(\frac{\eta}{2})) < \pi,$$

$$K(2 \arcsin(\frac{\eta + \epsilon}{2}) < \pi$$

for some $\epsilon > 0$. In this case, we compute that

$$\|\exp(ia) - 1\| = \|\exp(i\|a\|) - 1\| \le K(\eta + \epsilon). \qquad \text{(e 7.352)}$$

It follows that

$$\mathrm{dist}(\overline{u}, \overline{e}) < K(\eta + \epsilon) \qquad \text{(e 7.353)}$$

for all $\epsilon > 0$. Therefore

$$\mathrm{dist}(\overline{u}, \overline{e}) < K\eta. \qquad \text{(e 7.354)}$$

Otherwise, if $K\eta \ge 2$, we certainly have

$$\mathrm{dist}(\overline{u}, \overline{e}) \le 2 \le K\eta. \qquad \text{(e 7.355)}$$

Now we assume that $A \in \mathcal{C}_{1,1}$. If $\eta < 2$, by (e 7.350) and by 7.3,

$$\mathrm{cel}_{eAe}(u) \le K(\eta + \epsilon) + 8\pi \tag{e 7.356}$$

for all $\epsilon > 0$. It follows that

$$\mathrm{cel}_{eAe}(u) \le K\eta + 8\pi \tag{e 7.357}$$

If $\eta = 2$, choose $R = [\mathrm{cel}(w)] + 1$. Thus $\frac{\mathrm{cel}(w)}{R} < 1$. There is a projection $e' \in M_{R+1}(A)$ such that

$$[(1-e) + e'] = (K + RK)[e]. \tag{e 7.358}$$

It follows from 3.1 of [33] that

$$\mathrm{dist}(\overline{u + (1-e) + e'}, \overline{1 + e'}) < \frac{\mathrm{cel}(w)}{R+1}. \tag{e 7.359}$$

Put $K_1 = K(R+1)$. Then, from what we have proved,

$$\mathrm{cel}(u) \le K_1(\frac{\mathrm{cel}(w)}{R+1}) + 8\pi \tag{e 7.360}$$

$$= K\mathrm{cel}(w) + 8\pi. \tag{e 7.361}$$

□

8 $\mathcal{Z}$-stability

Definition 8.1. Let A be a unital separable C^*-algebra. We say A has finite weak tracial nuclear dimension if the following holds: For any $\epsilon > 0$, $\sigma > 0$, and any finite subset $\mathcal{F} \subset A$, there exists a projection $p \in A$ and a unital C^*-subalgebra B with $1_B = p$ and with $\dim_{nuc} B = m < \infty$ satisfying the following:

$$\|px - xp\| < \epsilon \text{ for all } x \in \mathcal{F}, \tag{e 8.362}$$

$$\dim(pxp, B) < \epsilon \text{ for all } x \in \mathcal{F}, \tag{e 8.363}$$

$$\tau(1-p) < \sigma \text{ for all } \tau \in T(A) \text{ and } T(A) \neq \emptyset. \tag{e 8.364}$$

The following is a based on a result of Winter ([60]).

Lemma 8.2. *Let A be a unital separable simple infinite dimensional C^*-algebra which has finite weak tracial nuclear dimension. Suppose that each unital hereditary C^*-subalgebra of A has the property of tracial m-almost divisible for some integer $m \ge 0$. Then, for any integer $k \ge 1$, there is a sequence of order zero contractive completely positive linear maps $L_n : M_k \to A$ such that $\{L_n(e)\}$ is central sequence of A for a minimal projection $e \in M_k$ and such that, for every integer $m \ge 1$,*

$$\lim_{n\to\infty} \max_{\tau \in T(A)} \{|\tau(L_n(e)^m) - 1/k|\} = 0. \tag{e 8.365}$$

Proof. Let $\{x_1, x_2, ...,\}$ be a dense sequence in the unit ball of A. Let $k \ge 1$ be an integer. Fix an integer $n \ge 1$. There is $1 > \gamma_n > 0$ such that

$$\|a^{m/n}x - xa^{m/n}\| < 1/4n, \;\; m = 1, 2, ..., n$$

for all $0 \le a \le 1$ and those x such that $\|x\| \le 1$ and

$$\|ax - xa\| < \gamma_n.$$

Since A has finite weak tracial nuclear dimension, there is a projection $p_n \in A$ and a unital C^*-subalgebra B_n with $1_B = p_n$ and with $\dim_{nuc} B = d_n < \infty$ satisfying the following:

$$\begin{aligned} \|p_n x_i - x_i p_n\| &< 1/4n, \ \ i = 1, 2, ..., n, & \text{(e 8.366)} \\ \dim(p_n x_i p_n, B_n) &< 1/3n, \ \ i = 1, 2, ..., n, \ \text{ and} & \text{(e 8.367)} \\ \tau(1 - p_n) &< 1/4n \ \text{ for all } \ \tau \in T(A). & \text{(e 8.368)} \\ & & \text{(e 8.369)} \end{aligned}$$

Let $y(i, n) \in B_n$ such that $\|y_{i,n}\| \le 1$ and

$$\|p_n x_i p_n - y(i, n)\| < 1/2n, \ \ i = 1, 2, ..., n. \qquad \text{(e 8.370)}$$

It follows from Lemma 4.11 of [60], since $p_n A p_n$ has the tracial m-almost divisible property, that there is an order zero contractive completely positive linear map $\Phi_n : M_k \to p_n A p_n$ such that

$$\begin{aligned} \|\Phi_n(a) y_i - y_i \Phi_n(a)\| &< \min\{1/4n, \gamma_n/4\}\|a\| \ \text{ for all } \ a \in A, \ \ i = 1, 2, ..., n, & \text{(e 8.371)} \\ \text{and } \ \tau(\Phi_n(\mathrm{id}_{M_k})) &\ge (1 - 1/n) \ \text{ for all } \ \tau \in T(p_n A p_n). & \text{(e 8.372)} \end{aligned}$$

By Proposition 1.1 of [60], there is a homomorphism $\varphi_n : C_0((0, 1]) \otimes M_k \to A$ such that $\Phi_n(a) = \psi_n(\imath \otimes a)$ for all $a \in M_k$, where $\imath(t) = t$ for $t \in (0, 1]$. Let $c_n = \imath^{1/n}$. Define $L_n(a) = \psi_n(c_n \otimes a)$. Clearly L_n is a zero order contractive completely positive linear map. It follows that

$$(L_n(e_1))^m = \psi_n(c_n^m \otimes e_1) = \psi_n((\imath \otimes e_1)^{m/n}) = (\Phi_n(e_1))^{m/n}.$$

Let $\{e_{i,j}\}$ be a matrix unit for M_k and denote by $e_i = e_{ii}$, $i = 1, 2, ..., k$. Let $x_{i,m} = \psi(c_n^{m/2} \otimes e_{i1})$. Then, since φ is a homomorphism,

$$x_{i,m} x_{i,m}^* = L_n(e_i)^m \ \text{ and } \ x_{i,m}^* x_{i,m} = L_n(e_1)^m, \ \ i = 1, 2, ..., k.$$

It follows that

$$\begin{aligned} \tau(L_n(e_1)^m) &= \tau(\varphi(c_n^m \otimes e_1)) & \text{(e 8.373)} \\ \ge \frac{1}{k}\tau(\Phi(\mathrm{id}_{M_k})) &\ge \frac{1 - 1/n}{k} \ \text{ for all } \ \tau \in T(p_n A p_n), & \text{(e 8.374)} \end{aligned}$$

$m = 1, 2, ..., n$. It follows that, for any $m \ge 1$,

$$\lim_{n\to\infty} \max_{\tau \in T(A)} |\tau(L_n(e_1)^m) - 1/k| = 0. \qquad \text{(e 8.375)}$$

We also have,

$$\begin{aligned} \|L_n(e_1) y_i - y_i L_n(e_1)\| &= \|\Phi_n(e_1)^{1/n} y_i - y_i \Phi_n(e_1)^{1/n}\| & \text{(e 8.376)} \\ &\le 1/4n, \ \ i = 1, 2, ..., n. & \text{(e 8.377)} \end{aligned}$$

It follows that

$$\|L_n(e_1) x_i - x_i L_n(e_1)\| < 1/n, \ \ i = 1, 2, ..., n. \qquad \text{(e 8.378)}$$

Since $\{x_1, x_2, ..., \}$ is dense in the unit ball of A, we conclude that

$$\lim_{n\to\infty} \|L_n(e_1) x - x L_n(e_1)\| = 0 \ \text{ for all } \ x \in A. \qquad \text{(e 8.379)}$$

So $\{L_n(e_1)\}$ is a central sequence of A. $\square$

We now apply the argument established in [39] to prove the following.

Theorem 8.3. *Let A be a unital separable simple C^*-algebra with finite weak tracial nuclear dimension. Suppose that A has the strict comparison property for positive elements and every unital hereditary C^*-subalgebra of A has the the property of m-almost divisible Cuntz semigroup. Then A is $\mathcal{Z}$-stable.*

Proof. By exactly the same argument for the proof that (ii) implies (iii) in [39], using 8.2 instead of Lemma 3.3 of [39], one concludes that any completely positive linear map from A into A can be excised in small central sequence. As in [39], this implies that A has property (SI). Using 8.2 instead of Lemma 3.3 of [39], the same proof that (iv) implies (i) in [39] shows that A is $\mathcal{Z}$-stable. □

Corollary 8.4. *Let A be a unital separable simple C^*-algebra in $\mathcal{C}_1$. Then A is $\mathcal{Z}$-stable.*

Proof. Note, by 6.6 and 4.1, that every unital hereditary C^*-subalgebra of A has the property of 0-almost divisible Cuntz semgroup. The lemma then follows from 8.3 and the fact that each C^*-algebra in $\mathcal{I}_d$ has nuclear dimension no more than d for all $d \geq 0$.

□

9 General Existence Theorems

Lemma 9.1. *Let X be a connected finite CW complex with $\mathrm{dim}X = 3$, let $Y = X \setminus \{\xi\}$, where $\xi \in X$ is a point, let Ω be a connected CW complex and let $\Omega' = \Omega \setminus \{\omega\}$. Let $P \in M_k(C(\Omega))$ (for some integer $k \geq 3(\mathrm{dim}Y + 1)$) be a projection with $\mathrm{rank}(P) \geq 3(\mathrm{dim}Y + 1)$. Let $\kappa \in KK(C_0(Y), C_0(\Omega'))$ such that $\kappa(F_3K_*(C_0(Y))) \subset F_3K_*(C_0(\Omega))$. Then there is a unital homomorphism $\varphi : C(X) \to PM_k(C(\Omega))P$ such that*

$$[\varphi|_{C_0(Y)}] = \kappa. \tag{e 9.380}$$

Proof. This is a combination of Proposition 3.16 and Theorem 3.10 of [15]. □

Corollary 9.2. *Let X be a connected finite CW complex such that $X^{(3)}$ (see 2.12) has r components, let $Y = X \setminus \{\xi\}$, where $\xi \in X$ is a point. Let Ω be a connected CW complex and let $\Omega' = \Omega \setminus \{\omega\}$ for some point $\omega \in \Omega$. Let $P \in M_k(C(\Omega))$ (for some integer $k \geq 3(\mathrm{dim}Y + 1)$) be a projection with $\mathrm{rank}(P) \geq 3(\mathrm{dim}Y + 1)$. Let $\kappa \in KK(C_0(Y), C_0(\Omega'))$ such that*

$$\kappa(F_3K_*(C_0(Y))) \subset F_3K_*(C_0(\Omega)) \text{ and } \kappa = \kappa_1 \circ s_3, \tag{e 9.381}$$

where $s_3 : C(X) \to C(X^{(3)})$ is the restriction and $\kappa_1 \in KK(C(X^{(3)}), C_0(\Omega'))$ such that $\kappa_1|_{\rho_{C(X^{(3)})}(C(X^{(3)})} = 0$ with the composition

$$K_0(C(X^{(3)}) = \mathrm{ker}\rho_{C(X^{(3)})} \oplus \rho_{C(X^{(3)})}(C(X^{(3)}).$$

Then there is a unital homomorphism $\varphi : C(X) \to PM_k(C(\Omega))P$ such that

$$[\varphi|_{C_0(Y)}] = \kappa. \tag{e 9.382}$$

Corollary 9.3. *Let X be a connected finite CW complex, let $Y = X \setminus \{\xi\}$, where $\xi \in X$ is a point. Suppose that $X^{(3)}$ has r components. Let Ω be a connected CW complex and let*

$\Omega' = \Omega \setminus \{\omega\}$ *for some point* $\omega \in \Omega$. *Let* $P \in M_k(C(\Omega))$ *(for some integer* $k \geq 3r(\mathrm{dim}Y + 1)$*) be a projection with* $\mathrm{rank}(P) \geq 3r(\mathrm{dim}Y + 1)$. *Let* $\kappa \in KK(C_0(Y), C_0(\Omega'))$ *such that*

$$\kappa(F_3K_*(C_0(Y))) \subset F_3K_*(C_0(\Omega)), \tag{e 9.383}$$

$$\kappa|_{F_mK_*(C_0(Y))} = 0 \text{ for all } m \geq 4 \text{ and} \tag{e 9.384}$$

$$\kappa|_{K_*(C_0(Y),\mathbb{Z}/k\mathbb{Z})} = 0 \text{ for all } k > 1. \tag{e 9.385}$$

Then there is a unital homomorphism $\varphi : C(X) \to PM_k(C(\Omega))P$ *such that*

$$[\varphi|_{C_0(Y)}] = \kappa. \tag{e 9.386}$$

Proof. Note that $\ker(s_3)_* = F_4K_*(C_0(Y))$. There is $\gamma \in Hom(K_*(C(X^{(3)})), K_*(C_0(\Omega')))$ such that $\gamma \circ (s_3)_*|_{K_*(C_0(Y))} = \kappa|_{K_*(C_0(Y))}$ and $\gamma|_{\rho_{C(X^{(3)})}(K_0(C(X^{(3)}))} = 0$. Define $\kappa_1|_{K_*(C(X^{(3)})} = \gamma$ and $\kappa_1|_{K_*(C(X^{(3)}),\mathbb{Z}/k\mathbb{Z}))} = 0$ for $k > 1$. Thus 9.2 applies. □

Lemma 9.4. *Let* X *be a connected finite CW complex with dimension* d *and with* $K_1(C(X)) = \mathbb{Z}^r \oplus \mathrm{Tor}(K_1(C(X))$, *let* $\delta > 0$, *let* $\mathcal{G} \subset C(X)$ *be a finite subset and let* $\mathcal{P} \subset \underline{K}(C_0(Z))$ *(where* $Z = X \setminus \{\xi\}$ *for some point* $\xi \in Z$*) be a finite subset. There exists an integer* $N_1(\delta, \mathcal{G}, \mathcal{P}) \geq 1$ *satisfying the following:*

Let Y *be a connected finite CW complex, and let* $\alpha \in KK(C_0(Z), C_0(Y_0)))$, *where* $Y_0 = Y \setminus \{y_0\}$ *for some point* $y_0 \in Y$. *For any projection* $P \in M_\infty(C(Y))$ *with* $\mathrm{rank}P \geq N_1(\delta, \mathcal{G}, \mathcal{P}) \cdot 3(\mathrm{dim}Y + 1)$, *there exists a unital* δ-$\mathcal{G}$*-multiplicative contractive completely positive linear map* $L : C(X) \to C = PM_R(C(Y))P$ *(for some integer* $R \geq \mathrm{rank}P$*) such that* $[L]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$.

Moreover, suppose that $\lambda : J_c(\mathbb{Z}^r)) \to U(M_d(PM_R(C(Y))P))/CU(M_d(PM_R(C(Y))P))$ *is a homomorphism with* $\Pi_c \circ \lambda \circ J_c = \alpha|_{K_1(C(X))}$ *(see 2.7), one may require that*

$$L^{\ddagger}|_{J(\mathbb{Z}^r)} = \lambda. \tag{e 9.387}$$

Proof. Note that $Tor(K_1(C(X)))$ is a finite group.

There is a connected finite CW complex T with $\mathrm{dim}T = 3$ such that $K_0(C_0(T_0)) = K_0(C_0(Z))$ and $K_1(C(T)) = Tor(K_1(C(X)))$, where $T_0 = T \setminus \{\xi\}$ and where $\xi \in T$ is a point.

Let $B = \overbrace{C(\mathbb{T}) \oplus C(\mathbb{T}) \oplus \cdots \oplus C(\mathbb{T})}^{k}$. We identify $\mathbb{Z}^r$ with $K_1(B)$. Let

$$B_0 = \overbrace{C_0(\mathbb{T}_0) \oplus C_0(\mathbb{T}_0) \oplus \cdots \oplus C_0(\mathbb{T}_0)}^{r},$$

where $\mathbb{T}_0$ is the circle minus a single point. Since $K_1(C(\mathbb{T}_0)) = K_1(C(\mathbb{T}))$, we obtain a isomorphism $\beta_1 : K_*(C_0(Z)) \to K_*(B_0 \oplus C_0(T_0))$. There is an invertible element $\beta \in KK(C_0(Z), B_0 \oplus C_0(T_0))$ such that $\beta|_{K_i(C_0(Z))} = \beta_1|_{K_i(C_0(Z))}$, $i = 0, 1$. There is, by [12], a unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map $L_1 : C(X) \to M_K(B \oplus C(T))$ such that

$$[L_1]|_{\mathcal{P}} = \beta|_{\mathcal{P}},$$

where K is an integer depending only on X, δ, $\mathcal{G}$ and $\mathcal{P}$. We may also assume that $L_1^{\ddagger}$ is defined and $L_1^{\ddagger}$ is invertible on $J_c(\mathbb{Z}^r)$. Put $N(\delta, \mathcal{G}, \mathcal{P}) = K + (rK)$.

Note that $\beta^{-1} \times \alpha \in KK(B_0 \oplus C_0(T_0), C_0(Y_0))$. There is $\alpha_1 \in KK(B_0, C_0(Y_0))$ and $\alpha_2 \in KK(C_0(T_0), C_0(Y_0))$ such that $\alpha_1 \oplus \alpha_2 = \beta^{-1} \times \alpha_0$.

Note that $\alpha_2(Tor(K_1(C(T)))) \subset Tor(K_1(C(Y)))$ (or it is zero). Note that $F_3(K_1(C(T))) = Tor(K_1(C(T)))$ and $F_m(K_1(C(T))) = \{0\}$ if $m > 3$ (see Lemma 3.3 of [17]). It follows from 2.13 that $Tor(K_1(C(Y))) \subset F_3(K_1(C(Y)))$. Therefore, by Theorem 3.10 and Proposition 3.16

of [15], there is a unital homomorphism $\varphi_1 : C(T) \to M_s(C(Y))$, where $s = 3(\text{dim}Y + 1)$ such that

$$[\varphi_1|_{C_0(T_0)}] = \alpha_2.$$

Note that $KK(B_0, C_0(Y_0)) = Hom(K_1(B_0), K_1(C_0(Y_0)))$. Let $z_j \in C(\mathbb{T})$ be the standard unitary generator of $C(\mathbb{T})$ for the j-th copy of $C(\mathbb{T})$ in B. Note that $K_1(B) = K_1(B_0)$ is generated by $[z_1], [z_2], ..., [z_r]$. We may assume that $[L_1] \circ J(\mathbb{Z}^k)$ is generated by $[z_j]$, $j = 1, 2, ..., r$.

Let $P \in M_R(C(Y))$ be a projection (for some large $R > 0$) such that

$$\text{rank}P \geq 6K(\text{dim}Y + 1).$$

Let Q be a trivial projection with rank Ks. We may assume that $P \geq Q$. Note also the map

$$\imath : U(M_s(C(Y))))/CU(M_s(C(Y)))) \to U(PM_R(C(Y))P)/CU(M_R(C(Y))P)$$

is an isomorphism. Let $\lambda_1 = \imath^{-1} \circ \lambda \circ (L_1^{\ddagger})^{-1}|_{J_c(\mathbb{Z}^r)}$. Put $u_j \in U(M_s(C(Y)))$ such that

$$[u_j] = \alpha_1([z_j]) \text{ and } \overline{u_j} = \lambda_1 \circ J([z_j]), \ \ j = 1, 2, ..., r. \qquad \text{(e 9.388)}$$

Define $\psi_j : C(\mathbb{T}) \to M_s(C(Y))$ by sending z_j to u_j, $j = 1, ..., r$. Put $\psi : B \to M_{ks}(C(Y))$ by $\psi = \text{diag}(\psi_1, \psi_2, ..., \psi_r)$. Define

$$L = (\text{diag}(\psi, \varphi_1) \otimes \text{id}_{M_K}) \circ L_1.$$

we check that L meets all requirements.

□

Lemma 9.5. *Let X, $\delta > 0$, $\mathcal{G} \subset C(X)$ and $\mathcal{P} \subset \underline{K}(C_0(Z))$ be as in 9.4. Let Y be a connected finite CW complex and let $\alpha \in KK(C_0(Z), C_0(Y_0)))$ such that*

$$\begin{aligned} \alpha(F_3K_*(C_0(Z)) &\subset F_3K_*(C_0(Y_0)), && \text{(e 9.389)}\\ \alpha|_{F_mK_*(C_0(Z))} &= 0 \ \textit{for all} \ m \geq 4 \ \text{and} && \text{(e 9.390)}\\ \alpha|_{K_*(C_0(Z), \mathbb{Z}/k\mathbb{Z})} &= 0 \ \textit{for all} \ k > 1, && \text{(e 9.391)} \end{aligned}$$

where $Y_0 = Y \setminus \{y_0\}$ for some point $y_0 \in Y$. Suppose, in addition, that

$$\lambda(J(\mathbb{Z}^r) \cap SU_d(C(X))/CU(M_d(C(X)))) \subset SU_d(C)/CU(M_d(C)),$$

where $C = PM_R(C(Y))P$. Then one may require L to be a homomorphism in the conclusion of Lemma 9.4 . Moreover, one can make $\text{rank}P = 6r_0r(d+1)$, *where r_0 is the number of connected components of $X^{(3)}$.*

Proof. Let r_0 be the number of connected components of $X^{(3)}$. By 9.3, there is a unital homomorphism $h_0 : C(X) \to M_{3r_0(d+1)}(C(Y))$ such that

$$[h_0] = \alpha|_{\underline{K}(C_0(Z))}.$$

We may write, as in 2.22, $J(\mathbb{Z}^r) = G_1 \oplus G_2$, where G_1 is free and $G_2 = J(\mathbb{Z}^r) \cap SU_d(C(X))$. Let $\lambda_1 = \lambda - h_0^{\ddagger}$. Since $\Pi_c \circ \lambda \circ J_c = \alpha|_{\mathbb{Z}^r}$, $\text{im}\,\lambda_1|_{J_c(\mathbb{Z}^r)} \subset U_0(M_d(C(Y))/CU(M_d(C(Y)))$. Thus, by the assumption,

$$\lambda_1(G_2) \subset (SU_d(C) \cap U_0(M_d(C)))/CU(M_d(C(Y))). \qquad \text{(e 9.392)}$$

It follows from 2.11 that $\lambda_1|_{G_2} = 0$. By 2.22, write $K_1(C(X^{(1)}) = S \oplus K_1(C(X))/F_3K_1(C(X))$. We may assume that $\Pi(G_1) = K_1(C(X))/F_3K_1(C(X))$, where $\Pi : U(M_d(C(X))/CU(M_d(C(X))) \to$

$K_1(C(X))$ is the quotient map. Let $G_1 = \mathbb{Z}^k$ be generated by the free generators $g_1, g_2, ..., g_k$ $(k \le r)$. Let $z_j \in M_d(C)$ be unitary such that $\bar{z}_j = \lambda_1(g_j)$, $j = 1, 2, ..., k$. Let $v_1, v_2, ..., v_k \in C(X)$ be unitaries representing $g_1, g_2, ..., g_k$, respectively. Since $X^{(1)}$ is 1-dimensional, it is easy and well known that there is a unital homomorphism $h_1 : C(X^{(1)}) \to M_{3k(d+1)}(C)$ such that $h_1(v_j) = z_j$, $j = 1, 2, ..., k$. Define $L = h_0 \oplus h_1$. □

Lemma 9.6. *Let X be a connected finite CW complex and let $Y = X \setminus \{\xi\}$, where $\xi \in X$ is a point. Let $K_0(C_0(Y)) = \mathbb{Z}^k \oplus Tor(K_0(C_0(Y)))$ and $K_1(C(X)) = \mathbb{Z}^r \oplus \mathrm{Tor}(K_1(C(X)))$. For any $\delta > 0$, any finite subset $\mathcal{G} \subset C(X)$ and any finite subset $\mathcal{P} \subset \underline{K}(C_0(Y))$, there exist integers $N_1, N_2 \ge 1$ satisfying the following:*

Let Ω be a finite CW complex and let $\kappa \in Hom_\Lambda(\underline{K}(C_0(Y)), \underline{K}(C(\Omega)))$ and let

$$K = \max\{|\rho_{C(\Omega)}(\kappa(g_i))| : g_i = (\overbrace{0, ..., 0}^{i-1}, 1, 0, ..., 0) \in \mathbb{Z}^k\}.$$

For any projection $P \in M_\infty(C(\Omega))$ with $\mathrm{rank}P \ge (N_2K + N_1(\mathrm{dim}Y + 1)\}$, there is a unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map $L : C(X) \to PM_N(C(\Omega))P$ (with some integer $N \ge \mathrm{rank}P$) such that

$$[L]|_{\mathcal{P}} = \kappa|_{\mathcal{P}}. \tag{e 9.393}$$

Moreover, if $\lambda : J(\mathbb{Z}^r) \subset J(K_1(C(X))) \to U(P(M_N(C(\Omega))P)/CU(PM_N(C(\Omega))P)$ is a homomorphism, then one may further require that

$$L^\ddagger|_{J(\mathbb{Z}^r)} = \lambda. \tag{e 9.394}$$

Proof. It suffices to show the case that Ω is connected. So in what follows we assume that Ω is a connected finite CW complex. Let $\omega \in \Omega$ be a point and let $\Omega_0 = \Omega \setminus \{\omega\}$. There is a splitting exact sequence

$$0 \to C_0(\Omega_0) \to C(\Omega) \to \mathbb{C} \to 0. \tag{e 9.395}$$

Thus

$$KK(C(X), C(\Omega)) = KK(C(X), \mathbb{C}) \oplus KK(C(X), C_0(\Omega_0)).$$

Write $\kappa = \kappa_0 \oplus \kappa_1$, where $\kappa_0 \in KK(C_0(Y), \mathbb{C})$ and $\kappa_1 \in KK(C_0(Y), C_0(\Omega_0))$.

Let $N_1(\delta/2, \mathcal{G}, \mathcal{P}) \ge 1$ be given by 9.4 for X, $\delta/2$, $\mathcal{G}$ and $\mathcal{P}$. Let $N_2 = N_2(\delta/2, \mathcal{G}, \mathcal{P})$ be the integer given by Lemma 10.2 of [34] for X, $\delta/2$, $\mathcal{G}$ and $\mathcal{P}$. There exists, by Lemma 10.2 of [34], a unital $\delta/2$-$\mathcal{G}$-multiplicative contractive completely positive linear map $L_1 : C(X) \to M_{N_2K}$ such that

$$[L_1](g_i) = \rho_{C(\Omega)}(\kappa_0(g_i)) \in \mathbb{Z}, \;\; [L_1]|_{\mathcal{P}} = \kappa_0|_{\mathcal{P}}. \tag{e 9.396}$$

Let $\imath : M_{N_2K} \to M_{N_2K}(C(\Omega))$ be a unital embedding. Put $N_1 = 3N_1(\delta/2, \mathcal{G}, \mathcal{P}) + 1$ and put $R_1 = N_1(\mathrm{dim}Y + 1)$. Let $P \in M_\infty(C(\Omega))$ be a projection whose rank is at least $N_2K + R_1$. Let $Q \le P$ be a trivial projection of rank N_2K. Let $P_1 = P - Q$. P_1 is a projection with at least rank R_1. Note that the embedding (for some large $R \ge 1$)

$$\gamma_1 : U(P_1M_R(C(\Omega))P_1)/CU(P_1M_R(C(\Omega))P_1) \to U(PM_R(C(\Omega))P)/CU(PM_R(C(\Omega))P)$$

is an isomorphism. Let

$$\gamma_2 : U(M_{N_2K}(C(\Omega)))/CU(M_{N_2K}(C(\Omega))) \to U(PM_R(C(\Omega))P)/CU(PM_R(C(\Omega))P).$$

be the homomorphism defined by $u \to P - Q + u$ for unitaries $u \in QM_R(C(\Omega))Q = M_{N_2K}(C(\Omega))$. Define $\lambda_1 = \gamma_1^{-1} \circ \lambda - \gamma_2 \circ \imath \circ L_1^{\ddagger}|_{J(\mathbb{Z}^r)}$. It follows from 9.4 that there is a $\delta/2$-$\mathcal{G}$-multiplicative contractive completely positive linear map $L_2 : C(X) \to P_1 M_R(C(\Omega))P_1$ such that

$$[L_2]|_{\mathcal{P}} = \kappa_1|_{\mathcal{P}} \text{ and } L_2^{\ddagger}|_{J(\mathbb{Z}^r)} = \lambda_1. \tag{e 9.397}$$

We then define

$$L = L_1 + L_2.$$

It is ready to verify that L, N_1 and N_2 meet all the requirements.

□

The following is a variation of a result of L. Li of [19].

Lemma 9.7. *Let X be a path connected compact metric space, let $\epsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)_{s.a.}$, there exists a unital homomorphism $\varphi_1 : C(X) \to C([0,1])$, an integer $N \geq 1$ satisfying the following, if $P \in M_r(C(Y))$ is a projection with*

$$\mathrm{rank}P(y) \geq N(\mathrm{dim}Y + 1) \text{ for all } y \in Y \tag{e 9.398}$$

and $\lambda : Aff(T(C(X))) \to Aff(T(PM_r(C(Y))P))$ is a unital positive linear map, where Y is a compact metric space, there is unital homomorphism $\varphi_2 : C([0,1]) \to PM_r(C(Y))P$ such that

$$|\tau \circ \varphi_2 \circ \varphi_1(f) - \lambda(f)(\tau)| < \epsilon \text{ for all } f \in \mathcal{F} \tag{e 9.399}$$

and for all $\tau \in T(PM_r(C(Y))P)$.

Proof. It follows from Lemma 2.9 of [19] that there exist a continuous map $\alpha : [0,1] \to X$ and a unital positive linear map $\gamma : C([0,1]) \to C(X)$ such that

$$|\tau(\gamma \circ (f \circ \alpha) - \tau(f)| < \epsilon/2 \text{ for all } f \in \mathcal{F} \tag{e 9.400}$$

and for all $\tau \in T(C(X))$. Let N be given by Corollary 2.6 of [19] for $X = [0,1]$, $\epsilon/2$ and for finite subset $\mathcal{G} = \{f \circ \alpha : f \in \mathcal{F}\}$. For $\lambda \circ \gamma$, by applying Corollary 2.6 of [19], one obtains a unital homomorphism $\varphi_2 : C([0,1]) \to PM_r(C(X))P$ such that

$$|\tau \circ \varphi_2(g) - \lambda \circ \gamma(g)(\tau)| < \epsilon/2 \text{ for all } g \in \mathcal{G} \tag{e 9.401}$$

and all $\tau \in T(PM_r(C(Y))P)$. Define $\varphi_1 : C(X) \to C([0,1])$ by $\varphi_1(f) = f \circ \alpha$ for all $f \in C(X)$. Then

$$|\tau(\varphi_2 \circ \varphi_1(f)) - \lambda(f)(\tau)| \leq |\tau \circ \varphi_2(f \circ \alpha) - \lambda \circ \gamma(f \circ \alpha)(\tau)| \tag{e 9.402}$$

$$+|\lambda \circ \gamma(f \circ \alpha)(\tau) - \lambda(f)(\tau)| \tag{e 9.403}$$

$$< \epsilon/2 + \epsilon/2 = \epsilon \tag{e 9.404}$$

for all $f \in \mathcal{F}$ and for all $\tau \in T(PM_r(C(Y))P)$.

□

Lemma 9.8. *Let $1 > \epsilon > 0$, Y be a finite CW complex, $r \geq 1$ be an integer and $C = PM_m(C(Y))P$ for some projection $P \in M_m(C(Y))$ such that $\mathrm{rank}\, P(y) \geq (6\pi/\epsilon)(\mathrm{dim}Y + 1)$ and $m \geq \mathrm{rank}P(y)$ for all $y \in Y$. Suppose that $u \in U(M_r(C))$ with $[u] = 0$ in $K_1(C)$ such that $u^k \in CU(M_r(C))$ for some integer $k \geq 1$, then*

$$\mathrm{dist}(\overline{u}, \overline{1_{M_r(C)}}) < \epsilon/r.$$

Proof. Since rank$P \ge 6\pi(\mathrm{dim}Y+1)$, $u \in U_0(M_r(C))$ (see [47]). Write $u = \prod_{j=1}^s \exp(\sqrt{-1}h_j)$, where $h_j \in M_r(C)_{s.a.}$, $j = 1, 2, ..., s$. Since $u^k \in CU(M_r(C))$,

$$\det(u^k(y)) = 1 \text{ for all } y \in Y.$$

It follows that

$$(\frac{1}{2\pi\sqrt{-1}})\sum_{j=1}^s Tr(h_j)(y) = I(y)/k \text{ for } y \in Y, \qquad \text{(e 9.405)}$$

where $I(y)$ is an integer for all $y \in Y$. Note that $I(y)/k$ is a continuous function on Y. Let $Y = Y_1 \sqcup Y_2 \sqcup \cdots \sqcup Y_l$, where each Y_i is connected, $i = 1, 2, ..., l$. It follows that $I(y)/k$ is a constant integer on each Y_i, $i = 1, 2, ..., l$. For each i, define f_i to be a number in $[-\pi, \pi]$ so that $\exp(\sqrt{-1}f_i) = \exp(\sqrt{-1}l(y)\pi/k)$ for $y \in Y$. Define

$$v(y) = \exp(-\sqrt{-1}f_i/r\mathrm{rank}P(y)) \text{ for all } y \in Y. \qquad \text{(e 9.406)}$$

It is a unitary in $U_0(M_r(C))$. Then

$$\|v - 1_{M_r(C)}\| < \epsilon/r. \qquad \text{(e 9.407)}$$

On the other hand,

$$\det((vu(y)) = 1 \text{ for all } y \in Y. \qquad \text{(e 9.408)}$$

Therefore, by 2.11, $vu \in SU_r(C) \cap U_0(M_r(C)) \subset CU(M_r(C))$. It follows from this and (e 9.407) that

$$\mathrm{dist}(\overline{u}, \overline{1_{M_r(C)}}) \le \mathrm{dist}(\overline{v}, \overline{1_{M_r(C)}}) < \epsilon/r. \qquad \text{(e 9.409)}$$

□

Theorem 9.9. *Let $X = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_s$ be a finite CW complex with dimension $d \ge 0$, where each X_i is a connected finite CW complex and let* $\ker\rho_{C(X)} = \mathbb{Z}^k \oplus Tor(K_0(C(X)))$. *Suppose that $\{g_1, g_2, ..., g_k\}$ is the standard generators for $\mathbb{Z}^k$.*

For any $\epsilon > 0$, any finite subset $\mathcal{G} \subset C(X)$, any finite subset $\mathcal{P} \subset \underline{K}(C(X))$, any finite subset $\mathcal{H} \subset C(X)_{s.a.}$, any $\sigma_1, \sigma_2 > 0$, any finite subset of $\mathcal{U} \subset U(M_d(C(X)))$, and any integer $S \ge 1$, there exists an integer N satisfying the following:

For any finite CW complex Y, any $\kappa \in Hom_\Lambda(\underline{K}(C(X)), \underline{K}(C(Y)))$ with $\kappa([1_{C(X_i)}]) = [P_i]$ for some projection $P_i \in M_m(C(Y))$ (and for some integer $m \ge 1$) , and $P = P_1 + P_2 + \cdots + P_s \in M_m(C(Y))$ is a projection, and rank$P_i(y) \ge \max\{NK,\, N(\mathrm{dim}Y+1)\}$ *for all $y \in Y$, where*

$$K = \max_{1 \le i \le k}\{\sup\{|\rho_{C(Y)}(\kappa(g_i))(\tau)| : \tau \in T(C(Y))\}\},$$

for any continuous homomorphism

$$\gamma : U(M_d(C(X)))/CU(M_d(C(X))) \qquad \text{(e 9.410)}$$
$$\to U(M_d(PM_m(C(Y))P))/CU(M_d(PM_m(C(Y))P)) \qquad \text{(e 9.411)}$$

and for any continuous affine map $\lambda : T(PM_m(C(Y)P) \to T(C(X))$ such that κ, γ and λ are compatible, then there exists a unital ϵ-$\mathcal{G}$-multiplicative contractive completely positive linear map $\Phi : C(X) \to PM_m(C(Y))P$ such that

$$[\Phi]|_{\mathcal{P}} = \kappa|_{\mathcal{P}}, \qquad \text{(e 9.412)}$$
$$\mathrm{dist}(\Phi^\ddagger(z), \gamma(z)) < \sigma_1 \textit{ for all } z \in \overline{\mathcal{U}} \text{ and} \qquad \text{(e 9.413)}$$
$$|\tau \circ \Phi(a) - \lambda(\tau)(a)| < \sigma_2 \textit{ for all } a \in \mathcal{H}. \qquad \text{(e 9.414)}$$

If $u_1, u_2, ..., u_m \in \mathcal{U}$ so that $[u_j] \neq 0$ in $K_1(C(X))$ and $\{[u_1], [u_2], ..., [u_m]\}$ generates a free group, we may require that

$$\Phi^{\ddagger}(\overline{u_j}) = \gamma(\overline{u_j}), \ \ j = 1, 2, ..., m. \tag{e 9.415}$$

Moreover, one may require that

$$P = Q_0 \oplus \mathrm{diag}(\overbrace{Q_1, Q_1, ..., Q_1}^{S_1}) \oplus Q_2$$

for some integer $S_1 \geq S$, where Q_0, Q_1 and Q_2 are projections in $PM_m(C(Y))P$, Q_0 is unitarily equivalent to Q_1, and $\Phi = \Phi_0 \oplus \overbrace{\Phi_1 \oplus \Phi_1 \oplus ... \oplus \Phi_1}^{S} \oplus \Phi_2$, $\Phi_0 : C(X) \to Q_0M_m(C(Y))Q_0$, $\Phi_1 = \psi_1 \circ h$, $\psi_1 : C(J) \to Q_1M_m(C(Y))Q_1$ is a unital homomorphism, $h : C(X) \to C(J)$ is a unital homomorphism, $\Phi_2 = \psi_2 \circ h$ and $\psi_2 : C(J) \to Q_2M_m(C(Y))Q_2$ is a unital homomorphism, where J is a disjoint union of s many unit intervals.

Proof. Without loss of generality, we may assume that $\mathcal{G}$ and $\mathcal{H}$ are in the unit ball of $C(X)$ and may assume that $\mathcal{H} \subset \mathcal{G}$. Moreover, to simplify notation, without loss of generality, we may also assume that X is connected. Write $K_1(C(X)) = \mathbb{Z}^{k_1} \oplus Tor(K_1(C(X)))$. Furthermore, we may assume that $\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \mathcal{U}_2$, where $\overline{\mathcal{U}_0} \subset \mathrm{Aff}(T(C(X)))/\overline{\rho_{C(X)}(K_0(C(X)))}$, $\overline{\mathcal{U}_1} \subset J_c(\mathbb{Z}^{k_1})$ and $\overline{\mathcal{U}_2} \subset J_c(Tor(K_1(C(X))))$. Let $\mathcal{H}_0 \subset C(X)_{s.a.}$ such that

$$\overline{\mathcal{U}_0} \subset \overline{\mathcal{H}_0}.$$

Put $\mathcal{H}_1 = \mathcal{H} \cup \mathcal{H}_0$. Let $0 < \delta < \epsilon$. We may also assume that $(2\delta, \mathcal{G}, \mathcal{P})$ is KK-triple. Choose an integer $R \geq 1$ such that

$$\frac{1}{RS} < \min\{\sigma_1/8\pi, \sigma_2/8\pi\}. \tag{e 9.416}$$

Let N_0 (in place of N) be the integer given by 9.7 for $\epsilon/4S$ (in place of ϵ) and $\mathcal{H}_1$ (in place of $\mathcal{F}$). Let N_1 and N_2 be integers given by 9.6 (for $\delta = \epsilon/2$). Put

$$N_1' = \max\{N_0, 2N_1, 2N_2, 48\pi/\sigma_1\}.$$

Define

$$N = 2(N_1' + 1)(2RS + 1).$$

Let $\kappa \in KK(C(X), C(Y))$ such that $\kappa([1_{C(X)}]) = P$ for some projection $P \in M_m(C(Y))$ (for some large $m \geq 1$) with $\mathrm{rank}P(y) \geq \max\{NK, N(\mathrm{dim}Y + 1)\}$, where

$$K = \max_{1 \leq i \leq k} \sup\{|\rho_{C(Y)}(\kappa(g_i))(\tau)| : \tau \in T(C(Y))\}.$$

To simplify the proof, by considering each connected component separately, we may assume that Y is connected. Let $Q_0 \in PM_m(C(Y))P$ be a projection with

$$\mathrm{rank}Q_0 = 2\max\{N_1K, N_1(\mathrm{dim}Y + 1)\} \geq N_1(K + \mathrm{dim}Y + 1).$$

This is possible because that $\mathrm{rank}P \geq \max\{NK, N(\mathrm{dim}Y + 1)\}$. Note that $P - Q$ has rank larger than $2N_1'RS(\mathrm{dim}Y + 1)$. Let $P_1 = P - Q_0$. There is $R_1 < N_1'\mathrm{dim}Y + 1)$ such that

$$\mathrm{rank}P_1 - (\mathrm{dim}Y + 1) = S_1N_1'(\mathrm{dim}Y + 1) + R_1 \tag{e 9.417}$$

for some integer $S_1 \geq RS$. Then there is a projection $P_1' \leq P_1$ such that P_1' is unitarily equivalent to

$$\mathrm{diag}(\overbrace{Q_0, Q_0, ..., Q_0}^{S_1}).$$

Write

$$P_1 = \overbrace{(Q_1, Q_1, ..., Q_1)}^{S_1} \oplus Q_2,$$

where Q_1 is unitarily equivalent to Q_0 and $Q_2 \lesssim Q_1$, where Q_2 has rank $R_1 + \mathrm{dim}Y + 1$. Note that

$$\frac{\mathrm{rank}Q_0 + \mathrm{rank}(Q_2)}{\mathrm{rank}P} < \min\{\sigma_1/4, \sigma_2/4\}. \tag{e 9.418}$$

For any integer $r \geq 1$, let $\Gamma : T(M_rC(Y)) \to T(PM_m(C(Y))P)$ be the map defined by

$$\Gamma(\tau)(a) = \int_Y tr(a) d\mu_\tau \tag{e 9.419}$$

for all $\tau \in T(C(Y))$ and all $a \in PM_m(C(Y))P$, where μ_τ is the probability measure induced by τ and where tr is the normalized trace on $M_{\mathrm{rank}P}$. Since the rank Q_1 is at least $N_0(\mathrm{dim}Y + 1)$, it follows from 9.7 that there is a unital homomorphism $h : C(X) \to C([0,1])$ and unital homomorphisms $\psi_1 : C([0,1]) \to Q_1M_m(C(Y))Q_1$ and $\psi_2 : C([0,1]) \to Q_2M_m(C(Y))Q_2$ such that

$$|\tau \circ \psi_1 \circ h(a) - \lambda \circ \Gamma(\tau)(a)| < \min\{\sigma_1, \sigma_2\}/2S \tag{e 9.420}$$

for all $a \in \mathcal{H}_1$ and all $\tau \in T(Q_1M_m(C(Y))Q_1)$. Define $\Phi_1 = \psi_1 \circ h$ and $\Phi_2 = \psi_2 \circ h$. Define $\Psi : C(X) \to P_1M_m(C(Y))P_1$ by $\Psi = \overbrace{(\Phi_1, \Phi_1, ..., \Phi_1)}^{S_1} \oplus \Phi_2$. Let $\kappa_1 = \kappa - [\Psi]$. Let

$$\Lambda : U(Q_0M_m(C(Y))Q_0)/CU(Q_0M_m(C(Y))Q_0) \to U(PM_m(C(Y))P)/CU(PM_m(C(Y))P)$$

be the isomorphism defined by $u \mapsto (P - Q_0) + u$ for all unitaries $u \in U(Q_0M_m(C(Y))Q_0)$. Define $\Lambda' : U(P_1M_m(C(Y))P_1)/CU(P_1M_m(C(Y))P_1) \to U(PM_m(C(Y))P)/CU(PM_m(C(Y))P)$ similarly. Define $\gamma_1 = \Lambda^{-1} \circ (\gamma - \Lambda' \circ \Psi^{\ddagger})$. Then, by applying 9.6, we obtain a unital $\epsilon/2$-$\mathcal{G}$-multiplicative contractive completely positive linear map $\Psi_1 : C(X) \to Q_0M_m(C(Y))Q_0$ such that

$$[\Psi_1]|_{\mathcal{P}} = \kappa_1|_{\mathcal{P}} \text{ and } \Psi_1^{\ddagger}|_{J(\mathbb{Z}^{k_1})} = \gamma_1|_{J(\mathbb{Z}^{k_1})}. \tag{e 9.421}$$

Put $C = PM_m(C(Y))P$. Define $\Phi = \Psi_1 \oplus \Psi$. It is clear that

$$[\Phi]|_{\mathcal{P}} = [\Psi_1]|_{\mathcal{P}} + [\Psi]|_{\mathcal{P}} \tag{e 9.422}$$

$$= \kappa_1|_{\mathcal{P}} + [\Psi]|_{\mathcal{P}} = \kappa|_{\mathcal{P}}. \tag{e 9.423}$$

It follows from (e 9.420) that

$$|\tau \circ \Phi(a) - \lambda(\tau)(a)| < \min\{\sigma_1/2, \sigma_2\} \tag{e 9.424}$$

for all $a \in \mathcal{H}_1$ and $\tau \in T(C)$. It follows from (e 9.421) that

$$\Phi^{\ddagger}(z) = \gamma(z) \text{ for all } z \in \overline{\mathcal{U}_1}. \tag{e 9.425}$$

Let $J_C : K_1(C(Y)) \to U(C)/CU(C)$ be the homomorphism defined in 2.7 which splits the following short exact sequence:

$$0 \to \mathrm{Aff}(T(C))/\overline{\rho_{C(Y)}(K_0(C))} \to U(C)/CU(C) \to K_1(C(Y)) \to 0. \qquad \text{(e 9.426)}$$

(It should be noted that $\mathrm{rank}P \geq N_1'(\mathrm{dim}Y + 1)$.) Let $z \in \overline{\mathcal{U}_2}$ and let $v_0, v_1 \in U(C)$ such that $\overline{v_0} = \lambda(z)$ and $\overline{v_1} = \Phi^{\ddagger}(z)$. Since γ is compatible with κ, we have

$$v_0^* v_1 \in U_0(C) \text{ and } (\lambda(z^*)\Phi^{\ddagger}(z))^k = \overline{1_C} \text{ in } U(C)/CU(C) \quad \text{for all } z \in \overline{\mathcal{U}_1}. \qquad \text{(e 9.427)}$$

Since $N_1 \geq 48\pi/\sigma_1$, by 9.8,

$$\mathrm{dist}(\lambda(z), \Phi^{\ddagger}(z)) < \sigma_1/2 \text{ for all } z \in \overline{\mathcal{U}_2}. \qquad \text{(e 9.428)}$$

Now for $z \in \overline{\mathcal{U}_0}$, by (e 9.425),

$$\mathrm{dist}(\Phi^{\ddagger}(z), \gamma(z)) < \sigma_1 \text{ for all } z \in \overline{\mathcal{U}_0}. \qquad \text{(e 9.429)}$$

The lemma follows. □

Corollary 9.10. *In the statement of 9.9, let $\xi_i \in X_i$ be a point and $X_i' = X_i \setminus \{\xi_i\}$, $i = 1, 2, ..., s$. Let $C = PM_m(C(Y))P$. Suppose that, in addition, Y is connected, $Y_0 = Y \setminus \{y_0\}$ for some point $y_0 \in Y$,*

$$\kappa|_{\underline{K}(C(X_i'))} \in KK(C_0(X_i'), C_0(Y_0)), \qquad \text{(e 9.430)}$$
$$\kappa(F_3K_*(C(X))) \subset F_3K_*(C), \quad \kappa|_{F_mK_*(C(X))} = 0 \textit{ for all } m \geq 4, \qquad \text{(e 9.431)}$$
$$\kappa|_{K_*(C(X), \mathbb{Z}/k\mathbb{Z})} = 0 \textit{ for all } k \geq 1 \text{ and} \qquad \text{(e 9.432)}$$
$$\lambda(SU_d(C(X))/CU(M_d(C(X)))) \subset SU_d(C)/CU(M_d(C)). \qquad \text{(e 9.433)}$$

Then Φ and Φ_0 can be required to be homomorphisms.

Proof. The proof is exactly same but applying 9.5 instead of 9.4. □

Corollary 9.11. *Let $\Omega = \Omega_1 \sqcup \Omega_2 \sqcup \cdots \sqcup \Omega_s$ be a disjoint union of connected finite CW complexes and let $X = \Omega \times \mathbb{T}$ has dimension $d+1$ and $X_j = \Omega_j \times \mathbb{T}$, $j = 1, 2, ..., s$. Then Lemma 9.9 holds for this X with d in the statement replaced by $d+1$ and with the following additional requirements. Suppose that $\mathcal{P} = \mathcal{P}_0 \sqcup \boldsymbol{\beta}(\mathcal{P}_1)$, where $\mathcal{P}_0, \mathcal{P}_1 \subset \underline{K}(C(\Omega))$ are finite subsets and suppose $\mathcal{U}_b \subset J_c(\boldsymbol{\beta}(F_2K_0(C(\Omega))) \cap \overline{\mathcal{U}}$ is a finite subset such that, in addition,*

$$\kappa|_{\boldsymbol{\beta}(\mathcal{P}_1)} = 0 \text{ and } \gamma|_{\mathcal{U}_b} = 0. \qquad \text{(e 9.434)}$$

Then, one can further require that

$$P_0 = P_{00} \oplus P_{01} \text{ and } \Phi_0 = \Phi_{00} \oplus \Phi_{01}, \qquad \text{(e 9.435)}$$

where $P_{00}, P_{0,1} \in M_m(C(Y))$ are projections, $\Phi_{00}(f \otimes g) = \sum_{j=1}^s f(\xi_j)q_j \cdot h_j(g)$ for all $f \in C(\Omega)$ and $g \in C(\mathbb{T})$, $\xi_j \in \Omega_j$ is a point, $q_{0,j} \in M_m(C(Y))$ is a projection with $\sum_{j=1}^s q_{0,j} = P_{00}$, $h_j : C(\mathbb{T}) \to q_{0,j}M_m(C(Y))q_{0,j}$ is a homomorphism $(j = 1, 2, ..., s)$ with $h_j(z) \in U_0(q_{0,j}M_m(C(Y))q_{0,j})$ $(j = 1, 2, ..., s)$, $\Phi_{01}(f \otimes g) = \sum_{j=1}^s L_j(f) \cdot g(1)q_{1,j}$ for all $f \in C(\Omega)$ and $g \in C(\mathbb{T})$, $1 \in \mathbb{T}$ is the point, $q_{1,j} \in M_m(C(Y))$ is a projection with $\sum_{j=1}^s q_{1,j} = P_{01}$, and $L_j : C(\Omega) \to q_{1,j}M_m(C(Y))q_{1,j}$ is a unital contractive completely positive linear map.

Proof. Note that, as in the proof of 9.9, one may assume that Ω is connected. Write $K_1(C(X)) = \mathbb{Z}^{k_1} \oplus Tor(K_1(C(X)))$. Note that

$$\boldsymbol{\beta}(F_2K_0(C(\Omega))) \subset F_3K_1(C(X)).$$

In that case, one may assume that

$$\mathcal{U} = \mathcal{U}_{00} \sqcup \mathcal{U}_{01} \sqcup \mathcal{U}_{01t} \sqcup \mathcal{U}_{11} \sqcup \mathcal{U}_b',$$

where

$$\overline{\mathcal{U}_{00}} \subset \mathrm{Aff}(T(C(X)))/\overline{\rho_{C(X)}(K_0(C(X))}, \tag{e 9.436}$$
$$\mathcal{U}_{01} \subset \{u \otimes 1_{C(\mathbb{T})} : u \in U(M_{3(d+1)}(C(\Omega))) \text{ and } [u] \in \mathbb{Z}^k \subset K_1(C(X))\}, \tag{e 9.437}$$
$$\mathcal{U}_{01t} \subset \{u \otimes 1_{C(\mathbb{T})} : u \in U(M_{3(d+1)}(C(\Omega))) \text{ and } [u] \in Tor(K_1(C(X))\} \tag{e 9.438}$$
$$\mathcal{U}_{11} = \{1 \otimes z\} \text{ and } \overline{\mathcal{U}_b'} = \mathcal{U}_b. \tag{e 9.439}$$

To simplify notation further, we may assume that $\mathcal{U}_b = \{x_1, x_2, ..., x_{n(b)}\}$, where $x_i \in U(M_{6(d+1)}(C(X)))$ such that there exists a unital homomorphism

$$H_i : M_2(C(S^3)) \to U(M_{6(d+1)}(C(X)))$$

such that $H_i(u_b) = x_i$, $i = 1, 2, ..., n(b)$. By the assumption, $\kappa([x_i]) = 0$, $i = 1, 2, ..., n(b)$. In the proof of 9.9, we choose R so that

$$1/RS < \min\{\sigma_1/16, \sigma_2/16\}.$$

We also choose Q_0 so that $\mathrm{rank}Q_0 = N_1K(\mathrm{dim}Y + 1) + 1$. Let $Q_{00} \leq Q_0$ have rank $N_1K(\mathrm{dim}Y)$ and Q_{01} have rank one. We proceed the proof and construct Φ_1, Φ_2 and Ψ. Since both Φ_1 and Φ_2 are homomorphisms which factor through $C(J)$, by 2.17,

$$\Psi^\ddagger(x) = 0 \text{ for all } x \in \mathcal{U}_b. \tag{e 9.440}$$

Let $\gamma_1 = \Lambda^{-1}(\gamma - \Lambda' \circ \Psi^\ddagger)$ (as γ_1 in the proof of 9.9). We then proceed to construct $L : C(\Omega) \to Q_{00}M_m(C(Y))Q_{00}$ the same way as Ψ_1 in the proof of 9.9 so that

$$[L]|_{\mathcal{P}_0} = \kappa_1|_{\mathcal{P}_0} \text{ and } (L)^\ddagger|_{\overline{\mathcal{U}_{01}}} = \gamma_1|_{\overline{\mathcal{U}_{01}}}. \tag{e 9.441}$$

Define $\Phi_{01} : C(X) \to Q_{00}M_m(C(Y))Q_{00}$ by $\Phi_{01}(f \otimes g) = L(f) \cdot g(1)Q_{00}$ for all $f \in C(\Omega)$ and $g \in C(\mathbb{T})$, where $1 \in \mathbb{T}$ is a point. Note that

$$\Phi_{01}^\ddagger(x) = 0 \text{ for all } x \in \mathcal{U}_b. \tag{e 9.442}$$

Now define $h : C(\mathbb{T}) \to Q_{01}M_m(C(Y))Q_{01}$ by $h(g) = g(x)$ for all $g \in C(\mathbb{T})$, where $x \in U_0(Q_{01}M_m(C(Y))Q_{01})$ is a unitary so that

$$\bar{x} = \gamma_1(\overline{1_{C(\Omega)} \otimes z}). \tag{e 9.443}$$

Define $\Phi_{00} : C(X) \to Q_{01}M_m(C(Y))Q_{01}$ by $\Phi_{00}(f \otimes g) = f(\xi)Q_{01} \cdot h(g)$ for all $f \in C(\Omega)$ and for all $g \in C(\mathbb{T})$, where $\xi \in \Omega$ is a point. We also have that

$$\Phi_{00}^\ddagger|_{\mathcal{U}_b} = 0. \tag{e 9.444}$$

We then define $\Phi_0 = \Phi_{00} \oplus \Phi_{01}$ and $\Phi = \Psi \oplus \Phi_0 \oplus \Psi \oplus \Phi_2$. As in the proof of 9.9, we have

$$\mathrm{dist}(\lambda(x), \Phi^\ddagger(x)) < \sigma_1 \text{ for all } x \in \overline{\mathcal{U}_{01t}}. \tag{e 9.445}$$

We have, for all $x \in \mathcal{U}$, as in the proof of 9.9,

$$\mathrm{dist}(\lambda(x), \Phi^{\ddagger}(x)) < \sigma_1 \ \text{for all} \ x \in \mathcal{U}. \tag{e 9.446}$$

The rest requirements now are ready to check.

□

Theorem 9.12. *Let X be a compact metric space such that $C(X) = \lim_{n\to\infty}(C(X_n), \psi_n)$, where each X_n is a finite CW complex and $\psi_n : C(X_n) \to C(X_{n+1})$ is a unital homomorphism and let $\varphi_{n,\infty} : C(X_n) \to C(X)$ be the unital homomorphism induced by the inductive limit system. For any $\epsilon > 0$, any finite subset $\mathcal{G} \subset C(X)$, any finite subset $\mathcal{P} \subset \underline{K}(C(X))$, any finite subset $\mathcal{H} \subset C(X)_{s.a.}$, any $\sigma_1, \sigma_2 > 0$, any finite subset $\mathcal{U} \subset U(M_r(C(X)))$ (for some integer $r \geq 1$) and any integer $L_1 \geq 1$, there exists an integer $n \geq 1$ such that $\mathcal{P} \subset [\varphi_{n,\infty}](\underline{K}(C(X_n))$, a finite set of mutually orthogonal projections $q_1, q_2, ..., q_s \in C(X_n)$ with $1_{C(X_n)} = q_1 + q_2 + \cdots + q_s$, a finite subset $g_1, g_2, ..., g_k$ which generates $\mathbb{Z}^k$ such that $\ker\rho_{K_0(C(X))} \cap \mathcal{P}$ is contained in a finitely generated subgroup $G_0 = \mathbb{Z}^k \oplus Tor(G_0)$, an integer $N \geq 1$ and a finitely generated subgroup $G_1 \subset U(M_l(C(X)))/CU(M_l(C(X)))$ (for some l) with $\overline{\mathcal{U}} \subset G_1$, $\Pi|_{G_1}$ is injective and $\Pi(G_1) \subset (\psi_{n,\infty})_{*1}(K_1(C(X_n)))$ satisfying the following:*

For any finite CW complex Y, any $\kappa \in Hom_{\Lambda}(\underline{K}(C(X_n)), \underline{K}(C))$ with $\kappa([q_i]) = [P_i]$ for some projection $P_i \in M_m(C(Y))$ (and for some integer $m \geq 1$), $P = P_1 + P_2 + \cdots + P_s \in M_m(C(Y))$ is a projection, $\mathrm{rank}P_i(y) \geq \max\{NK, N(\mathrm{dim}Y + 1)$ for all $y \in Y$ and for $C = PM_m(C(X))P$, where

$$K = \max_{1\leq i\leq k}\{\sup\{|\rho_C(\kappa(g_i'))(\tau)| : \tau \in T(C)\}\},$$

$g_i = (\psi_{n,\infty})_{*0}(g_i')$ for some $g_i' \in K_0(C(X_n))$, $i = 1, 2, ..., k$, for any continuous homomorphism

$$\gamma : G_1 + \mathrm{Aff}(T(C(X)))/\rho(K_0(C(X))) \tag{e 9.447}$$
$$\to U(M_l(C))/CU(M_l(C)) \tag{e 9.448}$$

and for any continuous affine map $\lambda : T(C) \to T(C(X))$ such that $\rho_{C(X)}([\psi_{n,\infty}(q_i)])(\lambda(\tau)) = \rho_C([P_i])(\tau)$ for all $\tau \in T(C)$, $\kappa([\psi_{n,\infty}](\xi)) = \Pi(\gamma(g))$ for all $g \in G_1$ and $\xi \in K_1(C(X_n))$ such that $[\psi_{n,\infty}](\xi) = g$, and λ and γ are compatible, then there exists a unital ϵ-$\mathcal{G}$-multiplicative contractive completely positive linear map $\Phi : C(X) \to PM_m(C(Y))P$ such that

$$[\Phi \circ \psi_{n,\infty}] = \kappa, \tag{e 9.449}$$
$$\mathrm{dist}(\Phi^{\ddagger}(x), \gamma(x)) < \sigma_1 \ \textit{for all} \ x \in \overline{\mathcal{U}} \ \text{and} \tag{e 9.450}$$
$$|\tau \circ \Phi(a) - \lambda(\tau)(a)| < \sigma_2 \ \textit{for all} \ a \in \mathcal{H}. \tag{e 9.451}$$

Moreover, one may require that

$$P = Q_0 \oplus \mathrm{diag}(\overbrace{Q_1, Q_1, ..., Q_1}^{L_1}) \oplus Q_2,$$

where Q_0, Q_1 and Q_2 are projections in $PM_m(C(Y))P$, Q_0 is unitarily equivalent to Q_1, and $\Phi = \Phi_0 \oplus \overbrace{\Phi_1 \oplus \Phi_1 \oplus ... \oplus \Phi_1}^{L_1} \oplus \Phi_2$, where $L_1 \geq L$ is an integer, $\Phi_0 : C(X) \to Q_0M_m(C(Y))Q_0$, $\Phi_1 = \psi_1 \circ \varphi_0$, $\psi_1 : C(J) \to Q_1M_m(C(Y))Q_1$ is a unital homomorphism, $\varphi_0 : C(X) \to C(J)$ is a unital ϵ-$\mathcal{G}$-multiplicative contractive completely positive linear map, $\Phi_2 = \psi_2 \circ \varphi_0$ and $\psi_2 : C(J) \to Q_2M_m(C(Y))Q_2$ is a unital homomorphism, where J is a disjoint union of s many unit intervals.

Proof. Fix $\epsilon > 0$ and a finite subset $\mathcal{G} \subset C(X)$. Without loss of generality, we may assume that $\mathcal{H} \subset \mathcal{G}$ and $\mathcal{G}$ are in the unit ball of $C(X)$.

We first prove the case that X is a compact subset of a finite CW complex of dimension d. Since we assume that, in this case, $\dim X_n = d$, the embedding from

$$U(M_d(C(X)))/CU(M_d(C(X)))$$

into $U(M_n(C(X))/CU(M_n(C(X)))$ is an isomorphism for all $n \geq d$. We may assume that $\mathcal{U} \subset U(M_d(C(X)))$, without loss of generality.

There is a sequence of decreasing finite CW complexes $\{X_n\}$ of dimension d such that $\cap_n X_n = X$. Write $C(X) = \lim_{n\to\infty}(C(X_n), r_n)$, where $r_n : C(X_n) \to C(X)$ is defined by $r_n(f) = f|_X$ for all $f \in C(X_n)$, $n = 1, 2, ...$ Let $(\delta, \mathcal{G}, \mathcal{P})$ be a KL-triple. We may assume that

$$\delta < \min\{\epsilon, \sigma_1/18(d+1)^2, \sigma_2/18(d+1)^2\}.$$

By 2.6 of [36], there exists an integer $n_0 \geq 1$ such that there is a unital $\delta/4$-$\mathcal{G}$-multiplicative contractive completely positive linear map $\Psi : C(X) \to C(X_{n_0})$ such that

$$\|r_{n_0} \circ \Psi(g) - g\| < \delta/4 \text{ for all } g \in \mathcal{G}. \tag{e 9.452}$$

We may assume that $\mathcal{P}' \subset \underline{K}(C(X_{n_0}))$ is a finite subset such that

$$[r_{n_0}](\mathcal{P}') = \mathcal{P}. \tag{e 9.453}$$

Suppose that $p, q \in M_m(C(X))$ are two projections such that $\tau \otimes Tr(p) = \tau \otimes Tr(q)$ for all $\tau \in T(C(X))$, where Tr is the standard trace on M_m. Then, $Tr(p(x)) = Tr(q(x))$ for all $x \in X$. With sufficiently large n_0, we may assume that there are projections $p', q' \in M_m(C(X_{n_0}))$ such that $p'|_X = p$ and $q'|_X = q$. Since $Tr(p)$ and $Tr(q)$ are integer values continuous functions on X_{n_0}, there is $n_0' \geq n_0$ such that

$$Tr(p'(x)) = Tr(q'(x)) \text{ for all } x \in X_{n_0'}. \tag{e 9.454}$$

Therefore, by choosing larger n_0, we may assume that

$$Tr(p'(x)) = Tr(q'(x)) \text{ for all } x \in X_{n_0}. \tag{e 9.455}$$

Let G_0' be the subgroup generated by $\ker\rho_{C(X)} \cap \mathcal{P}$. From the above (see (e 9.455)), we may assume, by choosing larger n_0, that

$$G_0' \subset (r_{n_0})_{*0}(\ker\rho_{C(X_{n_0})}). \tag{e 9.456}$$

We may write

$$\ker\rho_{C(X_{n_0})} = \mathbb{Z}^{k_1} \oplus \mathrm{Tor}(K_0(C(X_{n_0}))). \tag{e 9.457}$$

Let $s_1, s_2, ..., s_{k_1}$ be free generators of $\mathbb{Z}^{k_1}$. We may write that

$$(r_{n_0})_{*0}(s_i) = g_i, \;\; i = 1, 2, ..., k \text{ and } (r_{n_0})_{*0}(s_j) = 0 \text{ for all } j \geq k+1. \tag{e 9.458}$$

Thus we may write

$$G_0 = (r_{n_0})_{*0}(\ker\rho_{C(X_{n_0})}) = \mathbb{Z}^k \oplus \mathrm{Tor}(G_0). \tag{e 9.459}$$

We may also assume that $\mathcal{U}' \subset U(M_d(C(X)))$ such that $r_n(\mathcal{U}') = \mathcal{U}$. Let $\mathcal{G}' \subset C(X_{n_0})$ be a finite subset such that $r_n(\mathcal{G}') = \mathcal{G}$ and let $\mathcal{H}' \subset C(X_{n_0})_{s.a.}$ be a finite subset such that $r_n(\mathcal{H}') = \mathcal{H}$ and they are all in the unit ball of $C(X_{n_0})$. We may assume that, by (e 9.452),

$$[r_{n_0} \circ \Psi]|_{\mathcal{P}} = [\mathrm{id}_{C(X)}]|_{\mathcal{P}} \text{ and } (r_{n_0} \circ \Psi)^{\ddagger}|_{\mathcal{U}} = (\mathrm{id}_{C(X)})^{\ddagger}|_{\mathcal{U}}. \tag{e 9.460}$$

Suppose that $X_{n_0} = X_{n_0,1} \sqcup X_{n_0,2} \sqcup \cdots \sqcup X_{n_0,s}$ is a finite disjoint union of clopen subsets. Let $q_j = 1_{C(X_{n_0,j})}$, $j = 1, 2, ..., s$. Let $L_1 \geq 1$. Let $N \geq 1$ be given by 9.9 for X_{n_0} (in place of X), $\delta/4$ (in place of ϵ), $\mathcal{G}'$ (in place of $\mathcal{G}$), $\mathcal{P}'$ (in place of $\mathcal{P}$), $\mathcal{U}'$ (in place of $\mathcal{U}$), $\sigma_1/4$ (in place of σ_1) and $\sigma_2/4$ (in place of σ_2) and L_1. We choose $G_1 = J_c((r_{n_0})_{*1}(K_1(C(X_{n_0}))))$, where J_c may be chosen to be as in 2.15. Note that l can be chosen to be $\dim X_{n_0} + 1$.

Now suppose that κ is given as in the theorem (for the above g_i, $i = 1, 2, ..., k$, and q_j, $j = 1, 2, ..., s$) and $L_1 \geq 1$ is given.

By applying 9.9, there is a unital $\delta/4$-$\mathcal{G}'$-multiplicative contractive completely positive linear map $F : C(X_{n_0}) \to PM_m(C(Y))P$ such that

$$\begin{aligned} [F]|_{\mathcal{P}'} &= \kappa|_{\mathcal{P}'}, && \text{(e 9.461)}\\ \mathrm{dist}(F^{\ddagger}(z), \gamma \circ r_n^{\ddagger}(z)) &< \sigma_1/4 \text{ for all } z \in \overline{\mathcal{U}'} \text{ and} && \text{(e 9.462)}\\ |\tau \circ F(a) - \lambda(\tau)(r_{n_0})(a))| &< \sigma_2/4 \text{ for all } a \in \mathcal{H}', && \text{(e 9.463)} \end{aligned}$$

where $P = P_1 + P_2 + \cdots P_s$, $[P_j] = \kappa([q_j])$ and $\mathrm{rank} P_i(y) \geq \max\{NK, N(\dim Y + 1)\}$ for all $y \in Y$, $j = 1, 2, ..., s$. Moreover, as in the proof of 9.9, $P = Q_0 \oplus \mathrm{diag}(\overbrace{Q_1, Q_1, ..., Q}^{L_1} \oplus Q_2$, as required, and

$$F = \Psi_0 \oplus \overbrace{\Psi_1, \Psi_1, ..., \Psi_1}^{L_1} \oplus \Psi_2,$$

where $\Psi_0 : C(X_{n_0}) \to Q_0 M_m(C(Y))Q_0$ is a unital $\delta/4$-$\mathcal{G}'$-multiplicative contractive completely positive linear map $\Psi_1 = \psi_1' \circ h$ and $\Psi_2 = \psi_2' \circ h$, where $h : C(X_{n_0}) \to C(J)$ is a unital homomorphism and J is a disjoint union of finitely many intervals, $\psi_1' : C(J) \to Q_1 M_m(C(Y))Q_1$ and $\psi_2' : C(J) \to Q_2 M_m(C(Y))Q_2$ are unital homomorphisms.

Define $\Phi = F \circ \Psi$. It follows that

$$\begin{aligned} [\Phi]|_{\mathcal{P}} &= \kappa|_{\mathcal{P}}, && \text{(e 9.464)}\\ \mathrm{dist}(\Phi^{\ddagger}(z), \gamma(z)) &< \sigma_1 \text{ for all } z \in \overline{\mathcal{U}} \text{ and} && \text{(e 9.465)}\\ |\tau \circ F(a) - \lambda(\tau)(a)| &< \sigma_2 \text{ for all } a \in \mathcal{H}. && \text{(e 9.466)} \end{aligned}$$

For the general case, we may write that $C(X) = \overline{\cup_{n=1}^{\infty} C(X_n)}$, where each X_n is a compact subset of a finite CW complex. For any $\eta > 0$ and any finite subset $\mathcal{F} \subset C(X)$, we may assume that $\mathcal{F} \subset C(X_{n_1})$ for some $n_1 \geq 1$ with an error within $\eta/2$. Then, by 2.3.13 of [23], there is an integer $n_2 \geq 1$ and a unital $\eta/4$-multiplicative contractive completely positive linear map $\Psi' : C(X) \to C(X_{n_2})$ such that

$$\|\imath_{n_1} \circ \Psi'(f) - f\| < \eta/4 \text{ for all } f \in \mathcal{F}. \tag{e 9.467}$$

With sufficiently small η and large $\mathcal{F}$, by considering maps from $C(X_{n_2})$, one sees that the general case follows from the case that X is a compact subset of a finite CW complex.

□

Corollary 9.13. *Let Ω be a compact metric space and let $X = \Omega \times \mathbb{T}$. Then 9.12 holds for this X. Suppose also that $\mathcal{P} = \mathcal{P}_0 \sqcup \boldsymbol{\beta}(\mathcal{P}_1)$, where $\mathcal{P}_0$, $\mathcal{P}_1 \subset [\psi_{n,\infty}](\underline{K}(C(X_n)))$ are finite subsets and suppose that $\mathcal{U}_b \subset J_c(\boldsymbol{\beta}(F_2 K_0(C(\Omega)))) \cap \overline{\mathcal{U}}$ is a finite subset such that, in addition,*

$$\kappa|_{\boldsymbol{\beta}(\mathcal{P}_1)} = 0 \text{ and } \gamma|_{\mathcal{U}_b} = 0. \tag{e 9.468}$$

Then, one may further requite that

$$P_0 = P_{00} \oplus P_{01} \text{ and } \Phi_0 = \Phi_{00} \oplus \Phi_{01}, \tag{e 9.469}$$

where $P_{00}, P_{0,1} \in M_m(C(Y))$ *are projections,* $\Phi_{00}(f \otimes g) = \sum_{j=1}^{s} f(\xi_j)q_j \cdot h_j(g)$ *for all* $f \in C(\Omega)$ *and* $g \in C(\mathbb{T})$, $\xi_j \in \Omega_j$ *is a point,* $q_{0,j} \in M_m(C(Y))$ *is a projection with* $\sum_{j=1}^{s} q_{0,j} = P_{00}$, $h_j : C(\mathbb{T}) \to q_{0,j}M_m(C(Y))q_{0,j}$ *is a homomorphism with* $h_j(z) \in U_0(q_{0,j}M_m(C(Y))q_{0,j})$ $(j = 1, 2, ..., s)$, $\Phi_{01}(f \otimes g) = \sum_{j=1}^{s} L_j(f) \cdot g(1)q_{1,j}$ *for all* $f \in C(\Omega)$ *and* $g \in C(\mathbb{T})$, $1 \in \mathbb{T}$ *is a point,* $q_{1,j} \in M_m(C(Y))$ *is a projection with* $\sum_{j=1}^{s} q_{1,j} = P_{01}$, *and* $L_j : C(\Omega) \to q_{1,j}M_m(C(Y))q_{1,j}$ *is a unital contractive completely positive linear map.*

10 The uniqueness statement and the existence theorem for Bott map

The following is taken from 2.11 of [16].

Theorem 10.1. *Let* $\epsilon > 0$. *Let* $\Delta : (0, 1) \to (0, 1)$ *be an increasing map and let* $d \geq 0$ *be an integer. There exists* $\eta > 0$, $\gamma_1, \gamma_2 > 0$ *and a finite subset* $\mathcal{H} \subset C(\mathbb{T})_{s.a.}$ *and an integer* $N \geq 1$ *satisfying the following:*

Let $\varphi, \psi : C(\mathbb{T}) \to C = PM_r(C(Y))P$ *be two unital homomorphisms for some connected finite CW complex with* $\mathrm{dim}Y \leq d$ *and* $\mathrm{rank}P \geq N$ *such that*

$$|\tau \circ \varphi(g) - \tau \circ \varphi(g)| < \gamma_1 \text{ for all } g \in \mathcal{H} \text{ and for all } \tau \in T(C), \tag{e 10.470}$$

$$\mathrm{dist}(\varphi^{\ddagger}(\bar{z}), \varphi^{\ddagger}(\bar{z})) < \gamma_2, \tag{e 10.471}$$

$$\mu_{\tau\circ\varphi}(I_r) \geq \Delta(r) \text{ for all } \tau \in T(C) \tag{e 10.472}$$

and for all open arcs I_r *with length* $r \geq \eta$. *Then there exists a unitary* $u \in C$ *such that*

$$\|u^*\varphi(z)u - \psi(z)\| < \epsilon. \tag{e 10.473}$$

(Here $z \in C(\mathbb{T})$ is the identity map on the unit circle.)

Proof. The proof of 2.11 of [16] does not need the assumption that $\mathrm{dim}Y \leq 3$. The main technical lemma used in the proof was 4.47' of [18] which is a restatement of 4.47 which stated without assuming $\mathrm{dim}Y \leq 3$. Perhaps, a quick way to see this is to refer to the proof of Theorem 3.2 of [35] which is a modification of that of 2.11 of [16]. Again, note that Lemma 3.1 of [35] is another restatement of 4.47' of [18] which, as mentioned above, is a restatement of 4.47 of [18]. So Lemma 3.1 of [35] holds without assuming $d = \mathrm{dim}Y \leq 3$. However the integer L in Lemma 3.1 depends on d. There are two occasions that "since $\mathrm{dim}Y \leq 3$" appears in the proof of 3.2 of [35]. Both cases, we can simply replace 3 by d, in the next line (i.e., replace $3k_0m_1$ by dk_0m_1 and replace $3k_0l_1$ by dk_0l_1). Note also, since $X = \mathbb{T}$, $K_i(C(X))$ has no torsion. Therefore, we do not need D_j $(j \geq 2)$ in the modification. The same simple modification of proof of 2.11 of [16] also leads to this lemma.

□

Remark 10.2. Note that the above lemma also holds if φ and ψ are assumed to be unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps, where $\delta > 0$ and finite subset $\mathcal{G} \subset C(\mathbb{T})$ depend on ϵ, since $C(\mathbb{T})$ is weakly semi-projective.

Corollary 10.3. *Let X be a compact metric space, let $\mathcal{F} \subset C(X)$ be a finite subset, let $\epsilon > 0$ be a positive number and let $d \geq 1$. Let $\Delta : (0,1) \to (0,1)$ be a nondecreasing map. Let $\mathcal{U} \subset M_{m(X)}(C(X))$ be a finite subset of unitaries which represent non-zero elements in $K_1(C(X))$.*

There exists $\eta > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset C(X)$ and a finite subset $\mathcal{P} \subset \underline{K}(C(X))$, a finite subset $\mathcal{H} \subset C(X)_{s.a.}$, a finite subset $\mathcal{V} \subset K_1(C(X)) \cap \mathcal{P}$, an integer $N \geq 1$ satisfying the following: For any finite CW complex Y with $\mathrm{dim}Y \leq d$, any projection $P \in M_m(C(Y))$ with $\mathrm{rank}P(y) \geq N$ for all $y \in Y$ and two unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps $\varphi, \psi : C(X) \to C = PM_m(C(Y))P$ such that

$$[\varphi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}}, \tag{e 10.474}$$

$$\mu_{\tau\circ\varphi}(O_r) \geq \Delta(r), \;\; \mu_{\tau\circ\psi}(O_r) \geq \Delta(r), \tag{e 10.475}$$

for all $\tau \in T(M_n(C(Y)))$ and for all $r \geq \eta$,

$$|\tau\circ\varphi(g) - \tau\circ\psi(g)| < \gamma_1 \;\text{ for all } g \in \mathcal{H} \text{ and} \tag{e 10.476}$$

$$\mathrm{dist}(\overline{\langle\varphi(u)\rangle}, \overline{\langle\psi(u)\rangle}) < \gamma_2 \;\text{ for all } u \in J_{c(G(\mathcal{V})}(\mathcal{V}), \tag{e 10.477}$$

there exists, for each $v \in \mathcal{U}$, a unitary $w \in M_{m(X)}(C)$ such that

$$\|w(\varphi\otimes \mathrm{id}_{m(X)}(v)w^* - (\psi\otimes \mathrm{in}_{m(X)}(v)\| < \epsilon. \tag{e 10.478}$$

Proof. Fix $v \in \mathcal{U}$. Let $r \in (0,1)$. Choose a $r/2$-dense set $\{s_1, s_2, ..., s_n\}$ in $\mathbb{T}$. Let f_j be in $C(\mathbb{T})_+$ such that $0 \leq f_j \leq 1$, $f_j(s) = 1$ if $|s_j - s| \leq r/5$ and $f_j(s) = 0$ if $|s_j - s| \geq r/2$, $j = 1, 2, ..., n$.

Let $T_r = \{\tau \in T(C(X)) : \mu_\tau(O_r) \geq \Delta(r)\}$. It is easy to see that T_r is a compact subset of $T(C(X))$ (in the weak*-topology). Let

$$I_r = \{f \in C(X) : \tau(f^*f) = 0 \text{ for all } \tau \in T_r\}.$$

Then I_r is a closed two-sided ideal of $C(X)$. Put

$$J_r = \{f \in M_{m(X)}(C(X)) : (\tau\otimes Tr_{m(X)})(f^*f) = 0 \text{ for all } \tau \in T_r\}.$$

Note that $J_r \subset J_{r'}$ if $0 < r' < r$ and it is easy to check that $\cap_{1>r>0} J_r = \{0\}$. There is $1 > F_1(r) > 0$ such that

$$\pi(f_j(v)) \neq 0, \;\; j = 1, 2, ..., n, \tag{e 10.479}$$

where $\pi : M_{m(X)}(C(X)) \to M_{m(X)}(C(X))/J_{F_1(r)}$ is the quotient map. Therefore $\tau(f_j(v)) > 0$ for all $\tau \in T_{F_1(r)}$. Define

$$\Delta_v(r) = \inf\{\tau(f_j(v)) : \tau \in T_{F_1(r)}, j = 1, 2, ..., n\}. \tag{e 10.480}$$

Since $T_{F_1(r)}$ is compact, $\Delta_v(r) > 0$. We note that

$$\mu_\tau(I_r) \geq \Delta_v(r) \tag{e 10.481}$$

for all open arcs I_r. Define $\Delta_1 : (0,1) \to (0,1)$ by $\Delta_1(r) = \inf_{\{v\in\mathcal{U}\}} \Delta_v$. Then Δ_1 is an increasing map.

Now let $\eta_1 > 0$ (in place of η), $\gamma_1' > 0$ (in place of γ_1), $\gamma_2 > 0$, $\mathcal{H}_1$ (in place of $\mathcal{H}$) be a finite subset and $N \geq 1$ be an integer required by 10.1 for Δ_1 and ϵ given. Also let $\delta_1 > 0$ (in place of δ) and $\mathcal{G}_1$ (in place of $\mathcal{G}$) be finite subset given by 10.2 for ϵ.

Choose $1 > \eta > 0$ so that $\eta < F_1(\eta_1)$. Choose $\gamma_1 = \gamma_1'/m(X)$ and

$$\mathcal{H} = \{h_{i,i} \in C(X)_{s.a} : (h_{i,j}) = h(v) \quad \text{for} \quad \text{some } h \in \mathcal{H} \text{ and } v \in \mathcal{U}\}.$$

Choose $\delta = \delta_1/m(X)^2$ and

$$\mathcal{G} = \{g_{i,j} : (g_{i,j}) = g(v) \text{ for some } g \in \mathcal{G} \text{ and } v \in \mathcal{U}\}.$$

Choose a finite subset $\mathcal{P} \subset \underline{K}(C(X))$ so that it contains $[\mathrm{id}_{C(X)}] \in K_0(C(X))$ and $\{[v] \in K_1(C(X)) : v \in \mathcal{U}\}$. Choose $\mathcal{V} = \mathcal{U}$.

Now, if $\varphi, \psi : C(X) \to C = PM_m(C(Y))P$ are as described in the lemma, where $\mathrm{rank} P \geq N$, and $\dim Y = d$, which satisfy the assumption for the above chosen η, γ_1, γ_2, δ, $\mathcal{G}$, $\mathcal{H}$, $\mathcal{P}$, $\mathcal{V}$ and N, define $\lambda_v : C(\mathbb{T}) \to M_{m(X)}(C(X))$ by $\lambda_v(f) = f(v)$ for all $f \in C(\mathbb{T})$. Define $\varphi_v = (\varphi \otimes \mathrm{id}_{M_{m(X)}}) \circ \lambda_v$ and $\psi_v : (\psi \otimes \mathrm{id}_{M_{m(X)}}) \circ \lambda_v$. Then we apply 10.2 to φ_v and ψ_v. The lemma follows.

$\square$

10.4. *(Uniqueness statement for* $\dim Y \leq d$*) Let X be a compact metric space, let $\mathcal{F} \subset C(X)$ be a finite subset and let $\epsilon > 0$ be a positive number. Let $\Delta : (0,1) \to (0,1)$ be a nondecreasing map. There exists $\eta > 0$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\delta > 0$, a finite subset $\mathcal{G} \subset C(X)$ and a finite subset $\mathcal{P} \subset \underline{K}(C(X))$, a finite subset $\mathcal{H} \subset C(X)_{s.a.}$, a finite subset $\mathcal{V} \subset K_1(C(X)) \cap \mathcal{P}$, an integer $N \geq 1$ and an integer $K \geq 1$ satisfying the following: For any finite CW complex Y with* $\dim Y \leq d$, *any projection $P \in M_m(C(Y))$ with* $\mathrm{rank} P(y) \geq N$ *for all $y \in Y$ and two unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps $\varphi, \psi : C(X) \to C = PM_m(C(Y))P$ such that*

$$[\varphi]|_{\mathcal{P}} = [\psi]|_{\mathcal{P}}, \tag{e 10.482}$$

$$\mu_{\tau\circ\varphi}(O_r) \geq \Delta(r), \;\; \mu_{\tau\circ\psi}(O_r) \geq \Delta(r), \tag{e 10.483}$$

for all $\tau \in T(M_n(C(Y)))$ and for all $r \geq \eta$,

$$|\tau \circ \varphi(g) - \tau \circ \psi(g)| < \gamma_1 \text{ for all } g \in \mathcal{H} \text{ and} \tag{e 10.484}$$

$$\mathrm{dist}(\overline{\langle\varphi(u)\rangle}, \overline{\langle\psi(u)\rangle}) < \gamma_2 \text{ for all } u \in J_{c(G(\mathcal{V})}(\mathcal{V}) \tag{e 10.485}$$

(e 10.486)

there exists a unitary $W \in M_K(C)$ such that

$$\|W\varphi^{(K)}(f)W^* - \psi^{(K)}(f)\| < \epsilon \text{ for all } f \in \mathcal{F}, \tag{e 10.487}$$

where

$$\varphi^{(K)}(f) = \mathrm{diag}(\overbrace{\varphi(f), \varphi(f), ..., \varphi(f)}^{K}) \text{ and } \psi^{(K)}(f) = \mathrm{diag}(\overbrace{\psi(f), \psi(f), ..., \psi(f)}^{K})$$

for all $f \in C(X)$.

We actually will use a revised version of the above statement:

10.5. *The same statement 10.4 holds if (e 10.483) is replaced by*

$$\mu_{\tau\circ\varphi}(O_r) \geq \Delta(r) \text{ for all } \tau \in T(PM_m(C(Y))P). \tag{e 10.488}$$

By the virtue of 3.4 of [36], 10.5 and 10.4 are equivalent.

10.6. *(Existence statement for* dim$Y = d$*) Let X be a compact metric space with $C(X) = \lim_{n\to\infty}(C(X_n), \psi_n)$, where X_n is a finite CW complex, let $\mathcal{F} \subset C(X)$ be a finite subset, let $\epsilon > 0$ and $\delta_0 > 0$ be positive numbers. Let $\Delta : (0,1) \to (0,1)$ be a nondecreasing map. For any finite subset $\mathcal{P} \subset \underline{K}(C(X))$ and any finite subset $\mathcal{Q} \subset K_0(C(X))$, there exists $\eta > 0$, $\delta > 0$ and a finite subset $\mathcal{G} \subset C(X)$, an integer $n_0 \geq 1$, an integer $N \geq 1$ and an integer $K \geq 1$ satisfying the following: For any finite CW complex Y with* dim$Y \leq d$*, any unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map $\varphi : C(X) \to C = PM_R(C(Y))P$ (for some integer $R \geq 1$ and a projection $P \in M_R(C(X))$) such that* rank$P(y) \geq \max\{N, NK_1\}$ *for all $y \in Y$,*

$$\mu_{\tau\circ\varphi}(O_r) \geq \Delta(r) \tag{e 10.489}$$

for all balls O_r of X with radius $r \geq \eta$ and for all $\tau \in T(C)$, and for any $\kappa \in KK(C(X_{n_0}) \otimes C(\mathbb{T}), C(Y))$ with

$$\max\{|\rho_C(\kappa(\boldsymbol{\beta}(g_i')))(t)| : 1 \leq i \leq k \text{ and } t \in T(C)\} \leq K_1, \tag{e 10.490}$$

where $g_1, g_2, ..., g_k$ generates a subgroup which contains the subgroup generated by $K_1(C(X)) \cap \mathcal{P}$, $[\psi_{n_0,\infty}](g_i') = g_i$ for some $g_i' \in K_1(C(X_{n_0}))$, and, for any homomorphism

$$\Gamma : G(\mathcal{Q}) \to U(M_n(C))/CU(M_n(C(C))),$$

where $n = \max\{R(\boldsymbol{\beta}(G(\mathcal{Q}))), d+1\}$ (see 2.15*) and $\mathcal{Q} \subset [\psi_{n_0,\infty}](K_0(C(X_{n_0})$ with $\kappa \circ \boldsymbol{\beta}|_{\mathcal{Q}} = \Pi_C \circ \Gamma|_{\mathcal{Q}}$, there exists a unitary $u \in M_K(PM_R(C(Y))P)$ such that*

$$\begin{aligned} \|[\varphi^{(K)}(f), u]\| &< \epsilon \text{ for all } f \in \mathcal{F}, & \text{(e 10.491)}\\ \mathrm{Bott}(\varphi^{(K)} \circ \psi_{n_0,\infty}, u) &= K\kappa \circ \boldsymbol{\beta} \text{ and} & \text{(e 10.492)}\\ \mathrm{dist}(\mathrm{Bu}(\varphi^{(K)}, u)(x), K\Gamma(x)) &< \delta_0 \text{ for all } x \in \mathcal{Q}. & \text{(e 10.493)} \end{aligned}$$

The proof of 10.6 holds for dim$Y \leq d$ **under assumption that 10.4 holds for** dim$Y \leq d$.

Proof. We will apply 9.12 and the assumption that 10.4 holds for all finite CW complexes Y with dim$Y \leq d$. To simplify notation, we may assume, without loss of generality, that Y is connected.

We assume that $\epsilon < \delta_0/2$. We may assume that $1_{C(X)} \in \mathcal{F}$. Let $\Delta_1(r) = \Delta(r/3)/3$ for all $r \in (0,1)$. Let $\mathcal{P} \subset \underline{K}(C(X))$ be a finite subset. Let $(\epsilon', \mathcal{G}', \boldsymbol{\beta}(\mathcal{P}))$ be a KL-triple for $C(X) \otimes C(\mathbb{T})$. To simplify notation, by choosing a smaller ϵ and larger $\mathcal{F}$, we may assume that $(\epsilon, \mathcal{F}', \boldsymbol{\beta}(\mathcal{P}))$ is a KL-triple for $C(X) \otimes C(\mathbb{T})$, where $\mathcal{F}' = \{f \otimes g : f \in \mathcal{F} \text{ and } g \in \{1_{C(\mathbb{T})}, z, z^*\}\}$. To simplify notation, without loss of generality, we may also assume that $\mathcal{Q} = K_0(C(X)) \cap \mathcal{P}$.

There is $n_0' \geq 1$ such that $\mathcal{P} \subset [\psi_{n_0',\infty}](\underline{K}(C(X_{n_0'}))$. Let $\eta_1 > 0$ (in place of η) be as in 10.4 for $\epsilon/16$ and $\mathcal{F}$ and Δ_1. It follows from 3.4 of [36] that there is finite subset $\mathcal{H}_0 \subset C(X)_{s.a.}$ and $\sigma_0 > 0$ such that, for any unital positive linear maps $\psi_1, \psi_2 : C(X) \to C$ (for any unital stably finite C^*-algebra C),

$$|\tau(\psi_1(a)) - \tau(\psi_2(a))| < \sigma_0 \text{ for all } a \in \mathcal{H} \tag{e 10.494}$$

implies that

$$\mu_{\tau\circ\varphi_2}(O_r) \geq \Delta_1(r) \text{ for all } \tau \in T(C) \tag{e 10.495}$$

and for all open balls O_r of X with $r \geq \eta_1$, provided that

$$\mu_{\tau\circ\psi_1}(O_r) \geq \Delta(r) \tag{e 10.496}$$

for all open balls O_r with radius $r \geq \eta_1/4$ and for any tracial state τ of C.

Let $\eta = \eta_1/4$. Let $\gamma_1' > 0$ (in place γ_1) and $\gamma_2' > 0$ (in place γ_2) be as required by 10.4 for $\epsilon/16$ (in place of ϵ), X, Δ_1. Let $\sigma = \min\{\gamma_1'/4, \gamma_2'/4, \sigma_0, \delta_0/2\}$. Let $\delta_1 > 0$ (in place of δ), $\mathcal{G}_1 \subset C(X)$ (in place of $\mathcal{G}$) be a finite subset, $\mathcal{P}_1 \subset \underline{K}(C(X))$ be a finite subset, $\mathcal{H} \subset C(X)_{s.a.}$ be a finite subset, $\mathcal{V} \subset K_1(C(X)) \cap \mathcal{P}$, be a finite subset, N_1 (in place of N) be an integer and $K \geq 1$ be an integer as required by 10.4 for $\epsilon/16$ (in place of ϵ), $\mathcal{F}$, Δ_1 (and for X). We may assume that $\mathcal{F} \subset \mathcal{G}'$. Put $\mathcal{H}_1 = \mathcal{H} \cup \mathcal{H}_0$. Let $\mathcal{U}$ be a finite subset $U(M_L(C(X)))$ such that $\overline{\mathcal{U}} = J_{c(G(\mathcal{V})}(\mathcal{V})$. Let $G_{\mathcal{U}}$ be the subgroup of $K_1(C(X) \otimes C(\mathbb{T}))$ generated by $\{[u \otimes 1] : u \in \mathcal{U}\}$. Let $R(G_{\mathcal{U}})$ be as defined in 2.15. We may assume that $L \geq \max\{n, R(G(\mathcal{U}))\} \geq \dim Y + 1$.

Let n_0 (in place of n) be an integer, $q_1, q_2, ..., q_s \in C(X_{n_0} \times \mathbb{T})$ be a finite set of mutually orthogonal projections with $1_{C(X_{n_0} \times \mathbb{T})} = \sum_{j=1}^s q_j$ which represent each connected component of $X_{n_0} \times \mathbb{T}$, $s_1, s_2, ..., s_l$ (in place of $g_1, g_2, ..., g_k$), G_0 and $N_2 \geq 1$ (in place of N) be the integer required by 9.12 for $X \times \mathbb{T}$ (in place of X), $\epsilon/16$ (in place of ϵ), $\mathcal{G}_2 = \mathcal{G}' \otimes \{1_{C(\mathbb{T})}, z, z^*\}$ (in place of $\mathcal{G}$), $\mathcal{P}_1 \oplus \boldsymbol{\beta}(\mathcal{P})$ (in place of $\mathcal{P}$), $\mathcal{H}_1 \otimes \{1_{C(\mathbb{T})}\}$ (in place of $\mathcal{H}$), $\mathcal{U} \otimes \{1_{C(\mathbb{T})}\}$ (in place of $\mathcal{U}$), σ/L (in place of σ_1 and σ_2) and integer L_1. We may assume that $n_0 \geq n_0'$. Note that there are mutually orthogonal projections $q_1', q_2', ..., q_s' \in C(X_{n_0})$ such that $q_i = q_i' \otimes 1_{C(\mathbb{T})}$, $i = 1, 2, ..., s$. We may assume that

$$G_0 = \mathbb{Z}^{k_1} \oplus \mathbb{Z}^{k_2} \oplus \mathrm{Tor}(G_0), \qquad \text{(e 10.497)}$$

where $\mathbb{Z}^{k_1} \subset \ker\rho_{C(X)}$ and $\mathbb{Z}^{k_2} \in \boldsymbol{\beta}(K_1(C(X)))$. We may further write that $\mathrm{Tor}(G_0) = G_{00} \oplus G_{01}$, where $G_{00} \subset \ker\rho_{C(X)}$ and $G_{01} \subset \boldsymbol{\beta}(K_1(C(X)))$. We assume that $\mathbb{Z}^{k_1}$ is generated by $s_1, s_2, ..., s_{k_1}$ and $\mathbb{Z}^{k_2}$ is generated by $s_{k_1+1}, s_{k_1+2}, ..., s_{k_2}$. Write $s_{k_1+i} = \boldsymbol{\beta}(g_i)$, where $g_i \in K_1(C(X))$, $i = 1, 2, ..., k_2$. Without loss of generality, to simplify notation, we may assume that $G_0 \subset [\psi_{n_0,\infty}](K_0(C(X_{n_0}) \otimes C(\mathbb{T}))$, $\mathbb{Z}^{k_1} \in [\psi_{n_0,\infty}](K_0(C(X_{n_0})))$ and $\mathbb{Z}^{k_2} \subset \boldsymbol{\beta}([\psi_{n_0,\infty}](K_1(C(X_{n_0}))))$. Let $s_1', s_2', ..., s_{k_1}' \in K_0(C(X_{n_0}))$ and $g_{k_1+1}', g_{k_1+2}', ..., g_{k_1+k_2}' \in K_1(C(X_{n_0}))$ such that $[\psi_{n_0,\infty}](s_i') = s_i$, $i = 1, 2, ..., k_1$ and $[\psi_{n_0,\infty}](g_i') = g_i$, $i = 1, 2, ..., k_2$.

Let $N = \max\{N_1, N_2(\dim Y + 1)\}$. Let $\delta_2 > 0$ and $\mathcal{G}_3 \subset C(X)$ be a finite subset such that for any unital δ_2-$\mathcal{G}_3$-multiplicative contractive completely positive linear map $L' : C(X) \to C'$ (for any unital C^*-algebra with $T(C') \neq \emptyset$),

$$\tau([L'](s_j)) < 1/2N \ \text{ for all } \ \tau \in T(C'), \ \ j = 1, 2, ...k_1, \qquad \text{(e 10.498)}$$

(see 10.3 of [34]). We may assume that $\mathcal{F} \cup \mathcal{G}_1 \subset \mathcal{G}_3$. Let $G_4 = \{f \otimes g : f \in \mathcal{G}_2, g = 1_{C(\mathbb{T})}, z, z^*\}$. With even smaller δ_2, we may assume that for any unital δ_2-$\mathcal{G}_4$-multiplicative contractive completely positive linear map $L'' : C(X) \otimes C(\mathbb{T}) \to C'$, $[L'' \circ \psi_{n_0,\infty}]$ is well defined on $\underline{K}(C(X_{n_0}) \otimes C(\mathbb{T}))$.

Let $\delta = \min\{\delta_1, \delta_2\}$ and $\mathcal{G} = \mathcal{F} \cup \mathcal{G}_1 \cup \mathcal{G}' \cup \mathcal{G}_3 \subset C(X)$. Let $K_1 \geq 0$ be an integer. Let Y be a finite CW complex with dimension at most d, let $\varphi : C(X) \to C = PM_m(C(Y))P$ be a unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map with $\mathrm{rank}P \geq \max\{N, NK_1\}$ and

$$\mu_{\tau\circ\varphi}(O_r) \geq \Delta(r) \ \text{ for all } \ \tau \in T(C) \qquad \text{(e 10.499)}$$

and for all open balls O_r of X with radius $r \geq \eta$, let $\kappa \in KK(C(X_{n_0}) \otimes C(\mathbb{T}), C(Y))$ be such that

$$|\rho_C(\kappa \circ \boldsymbol{\beta}((g_i')))(t)| \leq K_1 \ \text{ for all } \ t \in T(C), \qquad \text{(e 10.500)}$$

$i = 1, 2, ..., k_2$. Note that, by (e 10.498), since $\mathrm{rank}P \geq NK_1$,

$$|\rho_C(\kappa(s_i'))(t)| \leq K_1 \ \text{ for all } \ t \in T(C), \qquad \text{(e 10.501)}$$

$i=1,2,...,k_1$. Without loss of generality, we may assume that $\varphi(\psi_{n_0,\infty}(q_j))=P_j$ is a projection, $j=1,2,...,s$, and $\sum_{j=1}^s P_j=\mathrm{id}_C$.

Define $\kappa_1\in Hom_\Lambda(\underline{K}(C(X_{n_0}\times\mathbb{T})),\underline{K}(C(Y))))$ by

$$\kappa_1|_{\underline{K}(C(X_{n_0}))}=[\varphi\circ\psi_{n_0,\infty}]\ \text{ and }\ \kappa_1|_{\boldsymbol{\beta}(\underline{K}(C(X_{n_0})))}=\kappa|_{\boldsymbol{\beta}(\underline{K}(C(X_{n_0})))}.$$

Let Γ be given as in the statement so that $\Pi_C\circ\Gamma|_{\mathcal{Q}}=\kappa_1\circ\boldsymbol{\beta}|_{\mathcal{Q}}$. Define $\lambda: T(C)\to T(C(X\times\mathbb{T}))$ by

$$\lambda(\tau)(f\otimes g)=\tau(\varphi(f))\cdot t_m(g)$$

for all $\tau\in T(C)$ and for all $f\in C(X)$ and $g\in C(\mathbb{T})$, where t_m is the tracial state of $C(\mathbb{T})$ induced by the normalized Lesbegue measure. One checks that λ is compatible with κ_1. Fix a splitting map $J_{M_L(C)}: K_1(C)\to U(M_L(C))/CU(M_L(C))$ (note that $L\geq \dim Y+1$), we write

$$U(M_L(C))/CU(M_L(C))=\mathrm{Aff}(T(M_L(C)))/\overline{\rho_{M_L(C)}(K_0(C))}\oplus J_{M_L(C)}(K_1(C)).$$

Let $G_1=G_{\mathcal{U}}+\boldsymbol{\beta}(G(\mathcal{Q}))$. Note since $K_1(C(X)\otimes C(\mathbb{T}))=K_1(C(X))\oplus\boldsymbol{\beta}(K_0(X))$, we may write $G_1=G_{\mathcal{U}}\oplus\boldsymbol{\beta}(G(\mathcal{Q})$. Note also that we have assumed that

$$G_1\subset[\psi_{n_0,\infty}](K_1(C(X_{n_0}))\oplus\boldsymbol{\beta}\circ[\psi_{n_0,\infty}](K_0(C(X_{n_0}))).$$

To simplify notation, we may assume that $G_1=[\psi_{n_0,\infty}](K_1(C(X_{n_0}))\oplus\boldsymbol{\beta}\circ[\psi_{n_0,\infty}](K_0(C(X_{n_0}))).$ We note that there is an injective homomorphism $J_c: G_1\to U(M_L(C(X)\otimes C(\mathbb{T})))/CU(M_L(C(X)\otimes C(\mathbb{T}))$ such that $\Pi\circ J_c=\mathrm{id}_{G_1}$ (see 2.15), where Π is defined in 2.15 for $C(X)\otimes C(\mathbb{T})$. Denote $\imath: U(M_n(C))/CU(M_n(C))\to U(M_L(C))/CU(M_L(C))$ the embedding given by $x\to 1_{L-n}\oplus x$. Define

$$\widetilde{\Gamma}: J(G_1)+\mathrm{Aff}(T(M_L(C(X\times\mathbb{T})))/\overline{\rho_{M_L(C(X\times\mathbb{T}))}(K_0(C(X\times\mathbb{T})))}\to U(M_L(C))/CU(M_L(C))$$

as follows. Let $\Gamma_1: G(\mathcal{Q})\to U(M_L(C))/CU(M_L(C))$ be defined by

$$\Gamma_1=\imath\circ\Gamma-J_{M_L(C)}\circ\kappa_1\circ\boldsymbol{\beta}|_{G(\mathcal{Q})}.$$

Since Γ and κ_1 are compatible, the image of Γ_1 is in $\mathrm{Aff}(T(M_L(C)))/\overline{\rho_{M_L(C)}(K_1(C))}$. Define

$$\begin{aligned}\widetilde{\Gamma}(x)&=\varphi^{\ddagger}(x)\ \text{ for all }\ x\in J(U_{\mathcal{U}}), &\text{(e 10.502)}\\ \widetilde{\Gamma}(x)&=\Gamma_1(\Pi(x))\ \text{ for all }\ x\in J(\boldsymbol{\beta}(G(\mathcal{Q}))\ \text{ and} &\text{(e 10.503)}\\ \widetilde{\Gamma}(x)&=\overline{\lambda}(x) &\text{(e 10.504)}\end{aligned}$$

for all $x\in\mathrm{Aff}(T(M_L(C(X\times\mathbb{T})))/\overline{\rho_{M_L(C(X\times\mathbb{T}))}(K_0(C(X\times\mathbb{T})))}$, where

$$\begin{aligned}\overline{\lambda}: Aff(T(M_L(C(X\times\mathbb{T}))))/\overline{\rho_{M_L(C(X\times\mathbb{T}))}(K_0(C(X\times\mathbb{T})))}\to& &\text{(e 10.505)}\\ Aff(T(M_L(C)))/\overline{\rho_{M_L(C)}(K_0(C))}& &\text{(e 10.506)}\end{aligned}$$

is the map induced by λ. Then $\widetilde{\Gamma}$ is continuous, moreover,

$$\kappa_1\circ\Pi|_{J([\psi_{n_0,\infty}](K_1(C(X_{n_0}\otimes C(\mathbb{T}))))}=\Pi_C\circ\tilde{\Gamma}|_{J([\psi_{n_0,\infty}](K_1(C(X_{n_0}\otimes C(\mathbb{T}))))},$$

$\lambda(\tau)(\psi_{n_0,\infty}(q_j))=\tau(P_j)$, $j=1,2,...,s$.

It follows from 9.12 that there is $\delta/4$-$\mathcal{G}$-multiplicative contractive completely positive linear map $\Phi: C(X)\to C$ such that

$$\begin{aligned}[][\Phi\circ\psi_{n_0,\infty}]&=\kappa_1 &\text{(e 10.507)}\\ \mathrm{dist}(\overline{\langle\Phi(x)\rangle},\widetilde{\Gamma}(x))&<\sigma/L\ \text{ for all }\ x\in\overline{\mathcal{U}\otimes 1_{C(\mathbb{T})}}\cup J(\boldsymbol{\beta}(\mathcal{Q}))\ \text{ and} &\text{(e 10.508)}\\ |\tau\circ\Phi(a)-\lambda(\tau)(a)|&<\sigma/L\ \text{ for all }\ a\in\mathcal{H}_1 &\text{(e 10.509)}\end{aligned}$$

and $\tau \in T(C)$. From (e 10.508), we also have

$$\mathrm{dist}(\overline{\langle \Phi(x) \rangle}, \Gamma(x)) \quad < \quad \sigma \ \text{for all} \ x \in J_{c(\boldsymbol{\beta}(G(\mathcal{Q})))}(\boldsymbol{\beta}(\mathcal{Q}))). \tag{e 10.510}$$

Let $\psi = \Phi|_{C(X)}$. Then

$$[\psi]|_{\mathcal{P}_1} = [\varphi]|_{\mathcal{P}_1}. \tag{e 10.511}$$

By (e 10.509) and the choices of $\mathcal{H}_0$ and σ_0, we have

$$\mu_{\tau\circ\psi}(O_r) \geq \Delta_1(r) \tag{e 10.512}$$

for all $\tau \in T(C)$ and for all open balls O_r with radius $r \geq \eta$. We also have

$$|\tau\circ\varphi(a) - \tau\circ\psi(a)| < \gamma_1' \ \text{for all} \ a \in \mathcal{H} \ \text{and} \tag{e 10.513}$$
$$\mathrm{dist}(\varphi^{\ddagger}(\bar{x}), \psi^{\ddagger}(\bar{x})) < \gamma_2' \ \text{for all} \ x \in \mathcal{U}. \tag{e 10.514}$$

It follows from 10.4 that there exists a unitary $W \in M_K(C)$ such that

$$\|\varphi^{(K)}(f) - W^*\psi^{(K)}(f)W\| < \epsilon/16 \ \text{for all} \ f \in \mathcal{F}. \tag{e 10.515}$$

Let v be a unitary in C such that

$$\|v - \Phi(1_{C(\mathbb{T})} \otimes z)\| < \epsilon/4. \tag{e 10.516}$$

Take $u = W^*\mathrm{diag}(\overbrace{v, v, ..., v}^{K})W$. Since we have assume that $(\epsilon, \mathcal{F}', \boldsymbol{\beta}(\mathcal{P}))$ is a KL-triple, this implies that

$$\mathrm{Bott}(\varphi^{(K)}\circ\psi_{n_0,\infty},\, u)|_{\mathcal{P}} = K\kappa\circ\boldsymbol{\beta}. \tag{e 10.517}$$

Moreover, by (e 10.510),

$$\mathrm{Bu}(\varphi^{(K)},\, u)(x), K\Gamma(J_c(\boldsymbol{\beta}(x))) < \sigma < \delta_0 \ \text{for all} \ x \in \mathcal{Q}. \tag{e 10.518}$$

□

11 The Basic Homotopy Lemma

In this section we will prove the following statement holds *under assumption that 10.4 holds for all finite CW complexes with* $\mathrm{dim}Y \leq d$.

11.1. *(Homotopy Lemma for* $\mathrm{dim}Y = d$*) Let X be a compact metric space, let $\epsilon > 0$, let $\mathcal{F} \subset C(X)$ be a finite subset and let $\Delta : (0,1) \to (0,1)$ be a nondecreasing map. There exists $\eta > 0, \delta > 0$, $\gamma > 0$, a finite subset $\mathcal{G} \subset C(X)$ and a finite subset $\mathcal{P} \subset \underline{K}(C(X))$, a finite subset $\mathcal{Q} \subset \mathrm{ker}\rho_{C(X)}$, an integer $N \geq 1$ and an integer $K \geq 1$ satisfying the following:*

Suppose that Y is a finite CW complex with $\mathrm{dim}Y \leq d$, *$P \in M_m(C(Y))$ is a projection such that* $\mathrm{rank}P \geq N$ *and $\varphi : C(X) \to C = PM_m(C(Y))P$ is a unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map and $u \in C$ is a unitary such that*

$$\|[\varphi(g),\, u]\| \quad < \quad \delta \ \textit{for all} \ g \in \mathcal{G}, \tag{e 11.519}$$
$$\mathrm{Bott}(\varphi,\, u)|_{\mathcal{P}} \quad = \quad 0, \tag{e 11.520}$$
$$\mathrm{dist}((\mathrm{Bu}(\varphi,\, u)(x), \bar{1}) \quad < \quad \gamma \ \textit{for all} \ x \in \mathcal{Q} \ \textit{and} \tag{e 11.521}$$
$$\mu_{\tau\circ\varphi}(O_r) \quad \geq \quad \Delta(r) \ \textit{for all} \ \tau \in T(PM_m(C(Y))P) \tag{e 11.522}$$

and for all open balls O_r of X with radius $r \geq \eta$. Then there exists a continuous path $\{u_t : t \in [0,1]\} \subset U(M_K(C))$ such that

$$u_0 = \mathrm{diag}(\overbrace{u, u, ..., u}^{K}), \quad u_1 = 1_{M_K(C)} \tag{e 11.523}$$

$$\text{and } \|[\varphi^{(K)}(f),\, u_t]\| < \epsilon \text{ for all } f \in \mathcal{F}. \tag{e 11.524}$$

11.2. *Let X be a compact metric space, let $\epsilon > 0$, let $\mathcal{F} \subset C(X)$ be a finite subset and let $\Delta : (0,1) \to (0,1)$ be a nondecreasing map. There exists $\eta > 0$, $\delta > 0$, $\gamma > 0$, a finite subset $\mathcal{G} \subset C(X)$ and a finite subset $\mathcal{P} \subset \underline{K}(C(X))$, a finite subset $\mathcal{Q} \subset \mathrm{ker}\rho_{C(X)}$, and an integer $N \geq 1$ and an integer $K \geq 1$ satisfying the following:*

Suppose that Y is a finite CW complex with $\mathrm{dim} Y \leq d$, $P \in M_m(C(Y))$ is a projection with $\mathrm{rank} P \geq N$ and $\varphi : C(X) \to C = PM_m(C(Y))P$ is a unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map and $u \in C$ is a unitary such that

$$\|[\varphi(c),\, u]\| < \delta \text{ for all } c \in \mathcal{G}, \tag{e 11.525}$$

$$\mathrm{Bott}(\varphi,\, u)|_{\mathcal{P}} = 0 \text{ and} \tag{e 11.526}$$

$$\mathrm{dist}(\mathrm{Bu}(\varphi,\, u)(x), \bar{1}) < \gamma \text{ for all } x \in \mathcal{Q}. \tag{e 11.527}$$

Suppose that there exists a contractive completely positive linear map $L : C(X) \otimes C(\mathbb{T}) \to C$ such that

$$\|L(c \otimes 1) - \varphi(c)\| < \delta, \ \|L(c \otimes z) - \varphi(c)u\| < \delta \text{ for all } c \in \mathcal{G} \tag{e 11.528}$$

$$\text{and } \mu_{\tau \circ L}(O_r) \geq \Delta(r) \text{ for all } \tau \in T(C) \tag{e 11.529}$$

and for all open balls O_r of $X \times \mathbb{T}$ with radius $r \geq \eta$, where $z \in C(\mathbb{T})$ is the identity function on the unit circle. Then there exists a continuous path $\{u_t : t \in [0,1]\} \subset U(M_K(C))$ such that

$$u_0 = u^{(K)} \ \ u_1 = 1_{M_K(C)} \tag{e 11.530}$$

$$\text{and } \|[\varphi^{(K)}(f),\, u_t]\| < \epsilon \text{ for all } f \in \mathcal{F}. \tag{e 11.531}$$

Proof. Note we assume that 10.4 holds for $\mathrm{dim} Y \leq d$.

Fix $1/2 > \epsilon > 0$ and $\mathcal{F}$ as stated in the 11.2. Without loss of generality, to simplify notation, we may assume that $\mathcal{F}$ is in the unit ball of $C(X)$. Let

$$\mathcal{F}' = \{f \otimes g : f \in \mathcal{F} \text{ and } g \in \{1, z, z^*\}\}.$$

Let $\eta > 0$, $\gamma_1, \gamma_2 > 0$, $\delta_1 > 0$ (in place of δ), $\mathcal{G}' \subset C(X \times \mathbb{T})$ (in place of $\mathcal{G}$) be a finite subset, $\mathcal{P}_1 \subset \underline{K}(C(X) \otimes C(\mathbb{T}))$ (in place of $\mathcal{P}$) be a finite subset, $\mathcal{H} \subset (C(X) \otimes C(\mathbb{T}))_{s.a.}$ be a finite subset and let $\mathcal{V} \subset K_1(C(X) \otimes C(T)) \cap \mathcal{P}_1$ be a finite subset, $N \geq 1$ be an integer and $K \geq 1$ be another integer required by 10.4 (in fact by 10.5) for $\epsilon/16$ (in place of ϵ), $\mathcal{F}'$ (in place of $\mathcal{F}$) and for $X \otimes \mathbb{T}$ (in place of X).

To simplify notation, without loss of generality, we may assume that

$$\mathcal{G}' = \{f \otimes g : f \in \mathcal{G}'' \text{ and } g \in \{1, z, z^*\}\},$$

where $\mathcal{G}'' \subset C(X)$ is a finite subset, $\mathcal{P}_1 = \mathcal{P}_0 \oplus \boldsymbol{\beta}(\mathcal{P})$, where $\mathcal{P}_0$, $\mathcal{P} \subset \underline{K}(C(X))$ are finite subsets and

$$\mathcal{H} = \{f \otimes g : f \in \mathcal{H}_0 \text{ and } g \in \mathcal{H}_1\},$$

where $1_{C(X)} \in \mathcal{H}_0 \subset C(X)_{s.a.}$ and $1_{C(\mathbb{T})} \in \mathcal{H}_1 \subset C(\mathbb{T})_{s.a.}$ are finite subsets. We may further assume that $\epsilon/16 < \delta_1$ and $\mathcal{F} \subset \mathcal{G}_1$ and $(2\delta_1, \mathcal{G}', \mathcal{P}_1)$ is a KL-triple.

Write $C(X) = \lim_{n\to\infty}(C(X_n), \psi_n)$, where each X_n is a finite CW complex and ψ_n is a unital homomorphism. We may assume that $n_0 \ge 1$ is an integer such that $\mathcal{F} \subset \psi_{n,\infty}(C(X_{n_0})$ and $\mathcal{P}_0, \mathcal{P} \subset [\psi_{n_0,\infty}](\underline{K}(C(X_{n_0}))$.

Suppose that $\mathcal{U} \subset U(M_R(C(X \otimes \mathbb{T})))$ is a finite subset such that $\overline{\mathcal{U}} = J_{c(G(\mathcal{V}))}(\mathcal{V})$ and $R = \max\{R(G(\mathcal{V})), d\}$. To simplify notation further, we may assume that $\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1$, where

$$\mathcal{U}_0 \subset \{u \otimes 1_{M_R} : u \in U(M_R(C(X))),\ [u] \neq 0 \text{ in } K_1(C(X))\}, \text{ and} \quad (\text{e}\,11.532)$$
$$\mathcal{U}_1 \subset \{x : [x] \in \boldsymbol{\beta}(K_0(C(X))) \text{ and } [x] \neq 0 \text{ in } K_1(C(X) \otimes C(\mathbb{T}))\}. \quad (\text{e}\,11.533)$$

Furthermore, we may write

$$G_{\mathcal{U}_1} = G_{00} \oplus G_{\mathbf{b}},$$

where $G_{\mathcal{U}_1}$ is the subgroup generated by $\{[u] : u \in \mathcal{U}_1\}$, G_{00} is a finitely generated free group such that $G_{00} \cap \boldsymbol{\beta}(\ker\rho_{C(X)}) = \{0\}$ and $G_{\mathbf{b}} \subset \boldsymbol{\beta}(\ker\rho_{C(X)})$ is also a finitely generated subgroup.

Let $G_{\mathcal{U}_0}$ be the subgroup of $K_1(C(X) \otimes C(\mathbb{T}))$ generated by $\{[u] : u \in \mathcal{U}_0\}$. Let $G(\mathcal{U}) = G_{\mathcal{U}_0} \oplus G_{\mathcal{U}_1}$. Accordingly, we may assume that

$$\overline{\mathcal{U}_1} = \overline{\mathcal{U}_{10} \sqcup \mathcal{U}_{11}},$$

where $\overline{\mathcal{U}_{10}}$ is a set of generators of $J_c(G_{00})$ and $\overline{\mathcal{U}_{11}}$ is the set of generators of $J_c(G_{\mathbf{b}})$. Put $\mathcal{U}_{\mathbf{b}} = \overline{\mathcal{U}_{11}}$. Note that $\mathcal{U}_{\mathbf{b}} = J_c(\boldsymbol{\beta}(F_2K_0(C(X)))\cap\overline{\mathcal{U}}$ and $J_c(G_{00}) \cap SU_R(C(X))/CU(M_R(C(X))) = \{0\}$. Let $\mathcal{Q} \subset \ker\rho_{C(X)}$ be a finite subset such that

$$\mathcal{Q} \supset \{x \in \ker\rho_{C(X)} : \boldsymbol{\beta}(x) = [u],\ \ u \in \Pi(\overline{\mathcal{U}_{11}})\}.$$

Let $\mathcal{G}_u \subset C(X)$ be a finite subset such that $\mathcal{U}_0 = \{(x_{ij})_{R\times R} : x_{ij} \in \mathcal{G}_u\}$. Let $0 < \delta_2 = \min\{\epsilon/16, \delta_1/16R^3, \gamma_1/16R^3, \gamma_1/16R^3\}$. Let
$dt_2 > dt_3 > 0$ be a positive number and $\mathcal{G} \supset \mathcal{G}'' \cup \mathcal{G}_u$ be a finite subset satisfying following: If $\varphi' : C(X) \to C'$ is a unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map, $u' \in C'$ is a unitary and $L' : C(X) \otimes C(\mathbb{T})$ is a unital contractive completely positive linear map, where C' is any unital C^*-algebra such that $\|[\varphi'(g), u']\| < \delta$ for all $g \in \mathcal{G}$,

$$\|\varphi'(g) - L'(g \otimes 1)\| < \delta \text{ and } \|\varphi'(g)u' - L'(g \otimes z)\| < \delta$$

for all $g \in \mathcal{G}$, then $[L' \circ \psi_{n_0,\infty}]$ is well defined on $\underline{K}(C(X_{n_0}) \otimes C(\mathbb{T}))$. In particular, we also assume that $[\psi' \circ \psi_{n_0,\infty}]$ is well defined on $\underline{K}(C(X_{n_0}))$. Let $\mathcal{G}_1 = \{f \otimes g : f \in \mathcal{G}, \text{ and } g = 1, z, z^*\}$. Without loss of generality, we may also assume that $(\Psi')^{\ddagger}$ is defined on $\mathcal{U}$ for any $2\delta_3$-$\mathcal{G}_1$-multiplicative unital contractive completely positive linear map $\Psi' : C(X) \otimes C(\mathbb{T}) \to C'$ for any unital C^*-algebra C'.

Let $\gamma = \min\{\gamma_1/16R^3, \gamma_2/16R^3\}$ and let $L_1 \ge 1$. Let $n \ge n_0$ be an integer, $q_1, q_2, ..., q_s \in C(X_n \times \mathbb{T})$ be mutually orthogonal projections with $1_{C(X\times\mathbb{T})} = \sum_{j=1}^s q_j$ which represent each connected component of X_n, $N \ge 1$ be an integer and G_1 be a finitely generated subgroup and $\mathcal{U}_{\mathbf{b}}$ be as required by 9.13 for $\epsilon/16$ (in place of ϵ), $\mathcal{G}$, G_0, $\mathcal{P}_1$ (in place of $\mathcal{P}$), $\mathcal{H}$ and γ (in place of σ_1 and σ_2). We may write $q_j = q_j' \otimes 1_{C(\mathbb{T})}$, where q_j' is a projection, $j = 1, 2, ..., s$. Let $\delta = \min\{\delta_1/2, \delta_2/2, \delta_3/2, \gamma/4\}$. Let Y be a finite CW complex with $\dim Y \le d$. Without loss of generality, to simplify notation, we may assume that Y is connected. Let $P \in M_m(C(Y))$ be a projection with $\mathrm{rank}P \ge N$ (so $\mathrm{rank}\,P \ge \max\{NK', N\}$, if $K' = 0$).

Suppose that $\varphi : C(X) \to C = PM_m(C(Y))P$ is a unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map and $L : C(X) \otimes C(\mathbb{T}) \to C$ is a unital δ-$\mathcal{G}_1$-multiplicative contractive completely positive linear map satisfying the assumption. Without loss of generality, we may assume that $L \circ \psi_{n,\infty}(q_j) = P_j = \psi \circ \psi_{n,\infty}(q_j)$ is a projection, $j = 1, 2, ..., s$ and

$1_C = \sum_{j=1}^s P_j$. Note, by the assumption above, $[\varphi\circ\psi_{n,\infty}]$ is well defined on $\underline{K}(C(X_{n_0}))$ and $[L\circ\psi_{n,\infty}]$ is well defined on $\underline{K}(C(X_{n_0})\otimes C(\mathbb{T}))$.

Let $H: C(X_n\times\mathbb{T})\to C$ defined by $H(f\otimes g)=\varphi(f)\cdot g(1)\cdot P$ for all $f\in C(X_n)$ and $g\in C(\mathbb{T})$, where $1\in\mathbb{T}$ is a point. Denote $\kappa=[H]$. Note that, by the assumption,

$$[L\circ\psi_{n,\infty}]=[H]. \tag{e 11.534}$$

Note also that $[H]|_{\boldsymbol{\beta}(\underline{K}(C(X_n)))}=0$. Let $\lambda: T(C)\to T(C(X\times\mathbb{T}))$ be defined by

$$\lambda(\tau)(g)=\tau\circ L(g)\ \text{ for all }\ g\in C(X\times\mathbb{T}) \tag{e 11.535}$$

and for all $\tau\in T(C)$. Note that $[L]|_{(\boldsymbol{\beta}(K_0(C(X)))\cap\mathcal{P})}=0$ Define

$$\Gamma_1: J_c(G(\mathcal{V}))+\mathrm{Aff}(T(M_R(C(X\times C(\mathbb{T})))))/\overline{\rho_{M_R(C(X\times\mathbb{T}))}(K_0(C(X\times\mathbb{T})))}\to U(M_R(C))/CU(M_R(C))$$

by

$$\begin{aligned}
\Gamma_1(z) &= \varphi^{\ddagger}(z)\ \text{ for all }\ z\in J_c(G(\mathcal{U}_0)), & \text{(e 11.536)}\\
\Gamma_1(z) &= L^{\ddagger}(z)\ \text{ for all }\ z\in J_c(G_{00}), & \text{(e 11.537)}\\
\Gamma_1(z) &= \bar{1}\ \text{ for all }\ z\in J_c(G_{\mathbf{b}})\ \text{ and} & \text{(e 11.538)}\\
\Gamma_1(z) &= \overline{\lambda}(z) & \text{(e 11.539)}
\end{aligned}$$

for all $z\in \mathrm{Aff}(T(M_R(C(X)\otimes C(\mathbb{T}))))/\overline{\rho_{M_R(C(X)\otimes C(\mathbb{T}))}(K_0(C(X)\otimes C(\mathbb{T})))}$, where $\overline{\lambda}$ is induced by λ. Note that $\Gamma_1|_{\mathcal{U}_{\mathbf{b}}}=0$. It is also clear that $\lambda(\tau)(q_j)=\tau(P_j)$, $j=1,2,...,s$.

It follows from 9.13 that there exist three unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps $\Phi_{00}: C(X)\otimes C(\mathbb{T}))\to P_{00}M_m(C(Y))P_{00}$, $\Phi_{01}: C(X)\otimes C(\mathbb{T})\to P_{01}M_m(C(Y))P_{01}$ and $\Phi_1: C(X)\otimes C(\mathbb{T})\to P_1M_m(C(Y))P_1$ with $P=P_{00}\oplus P_{01}\oplus P_1$, $\Phi_{00}(f\otimes g)=\sum_{j=1}^s f(\xi_j)e_{0,j}\cdot h_j(g)$ for all $f\in C(X)$ and $g\in C(\mathbb{T})$, where $\xi_j\in X$, $e_{0,j}\in P_{00}M_m(C(Y))P_{00}$, is a projection so that $\sum_{j=1}^s e_{0,j}=P_{00}$, $h_j: C(\mathbb{T})\to e_{0,j}M_m(C(Y))e_{0,j}$ is a unital homomorphism with $h_j(z)\in U_0(e_{0,j}M_m(C(Y))e_{0,j})$ $(j=1,2,...,s)$, $\Phi_{01}(f\otimes g)=\sum_{j=1}^s\Psi_j(f)\cdot g(1)\cdot e_{1,j}$ for all $f\in C(X)$ and $g\in C(\mathbb{T})$, where $1\in\mathbb{T}$ is the point on the unit circle, where $\Psi_j: C(X)\to e_{1,j}M_m(C(Y))e_{1,j}$ is a unital contractive completely positive linear map and $e_{1,j}\in P_{01}M_m(C(Y))P_{01}$ is a projection so that $\sum_{j=1}^s e_{1,j}=P_{01}$, $\Phi_1=\psi_1\circ\psi_0$, where $\psi_0: C(X\otimes\mathbb{T})\to C(J)$ is a δ-$\mathcal{G}$-multiplicative contractive completely positive linear map, J is a finite disjoint union of intervals, and $\psi_1: C(J)\to P_1M_m(C(Y))P_1$ is a unital homomorphism, such that

$$\begin{aligned}
[(\Phi_0\oplus\Phi_1)\circ\psi_{n,\infty}] &= [H], & \text{(e 11.540)}\\
\mathrm{dist}((\Phi_0\oplus\Phi_1)^{\ddagger}(z),\Gamma_1(z)) &< \gamma\ \text{ for all }\ z\in\mathcal{U}\ \text{ and} & \text{(e 11.541)}\\
|\tau\circ(\Phi_0\oplus\Phi_1)(a)-\lambda(\tau)(a)| &< \gamma\ \text{ for all }\ a\in\mathcal{H}\ \text{ and }\ \text{ for all }\ \tau\in T(C), & \text{(e 11.542)}
\end{aligned}$$

where $\Phi_0=\Phi_{00}\oplus\Phi_{01}$. Let $z_{00}=\Phi_{00}(1\otimes z)=\sum_{j=1}^s h_j(z)$. Since $h_j(z)\in U_0(e_{0,j}M_m(C(Y))e_{0,j})$, there is a continuous path of unitaries $\{z_{00,j}(t): t\in[1/2,1]\}\subset U_0(e_{0,j}M_m(C(Y))e_{0,j})$ such that

$$z_{00,j}(1/2)=h_j(z)\ \text{ and }\ z_{00,j}(1)=e_{0,j},\ \ j=1,2,...,s. \tag{e 11.543}$$

Define $z_{00}(t)=\sum_{j=1}^s z_{00,j}(t)$ for $t\in[0,1/2]$. Then

$$\Phi_{00}(f\otimes 1)z_{00}(t)=z_{00}(t)\Phi_{00}(f\otimes 1)\ \text{ for all }\ f\in C(\mathbb{T})\ \text{ and }\ t\in[1/2,1]. \tag{e 11.544}$$

Let $z_0=\psi_0(1\otimes z)$. There is a unitary $v_0\in C(J)$ such that

$$\|z_0-v_0\|<\delta. \tag{e 11.545}$$

There exists a continuous path of unitaries $\{w(t) : t \in [1/2, 1]\} \subset C(J)$ such that

$$w(1/2) = v_0, \ \ w(1) = 1. \tag{e 11.546}$$

Note that

$$w_t\varphi_0(f) = \varphi_0(f)w_t \ \text{ for all } \ f \in C(X) \ \text{ and } \ \ \text{for all } \ t \in [1/2, 1]. \tag{e 11.547}$$

On the other hand, we also have, by (e 11.534) , (e 11.540), the assumption (e 11.527) and (e 11.535),

$$\begin{aligned} [L]|_{\mathcal{P}_1} &= [\Phi_0 \oplus \Phi_1]|_{\mathcal{P}_1}, & \text{(e 11.548)} \\ \mathrm{dist}(L^{\ddagger}(x), \Gamma_1(x)) &< 2\gamma \ \text{ for all } \ x \in \mathcal{U} \ \text{ and} & \text{(e 11.549)} \\ |\tau(L(a)) - \lambda(\tau)(a)| &= 0 \ \text{ for all } \ g \in C(X) \otimes C(\mathbb{T}) \ \text{ and } \ \tau \in T(C). & \text{(e 11.550)} \end{aligned}$$

With these and (e 11.529) as well as (e 11.540), (e 11.541) and (e 11.542), we conclude, by applying 10.5, that there exists a unitary $W \in M_K(PM_m(C(Y))P)$ such that

$$\begin{aligned} \|L^{(K)}(f \otimes 1) - W^*(\Phi_0^{(K)} \oplus \Phi_1^{(K)})(f \otimes 1)W\| &< \epsilon/16 \ \text{ for all } \ f \in \mathcal{F} \ \text{ and } & \text{(e 11.551)} \\ \|u^{(K)} - W^*(z_{00}^{(K)} \oplus P_{01}^{(K)} \oplus v_0^{(K)})W\| &< \epsilon/16 + 2\delta < 3\epsilon/16, & \text{(e 11.552)} \end{aligned}$$

There exists a continuous path of unitaries $\{u_t : t \in [0, 1/2]\} \subset M_K(PM_m(C(Y))P)$ such that

$$\begin{aligned} & u_0 = u^{(K)}, \ \ u_{1/2} = W^*(z_{00}^{(K)} \oplus P_{01}^{(K)} \oplus v_0^{(K)})W \ \text{ and} & \text{(e 11.553)} \\ & \|u^{(K)} - u_t\| < \epsilon/2 \ \text{ for all } \ t \in [0, 1/2]. & \text{(e 11.554)} \end{aligned}$$

Define

$$u_t = W^*(z_{00}(t)^{(K)} \oplus P_{01}^{(K)} \oplus \psi_1(w(t)^{(K)})W \ \text{ for all } \ t \in [1/2, 1]. \tag{e 11.555}$$

Then $\{u_t : t \in [0, 1]\}$ is a continuous path of unitaries, $u_0 = u, \ \ u_1 = 1$ and

$$\begin{aligned} \|\varphi^{(K)}(f)u_t - u_t\varphi^{(K)}(f)\| &\le 2\delta + 2\|\varphi(f) - L(f \otimes 1)\| & \text{(e 11.556)} \\ &+ \|[L^{(K)}(f \otimes 1)u_t - u_tL^{(K)}(f \otimes 1)]\| & \text{(e 11.557)} \\ &< 6\delta < 3\epsilon/8 \quad \text{for all } \ f \in \mathcal{F} \ \text{ and } \quad \text{for all } \ t \in [0, 1]. & \text{(e 11.558)} \end{aligned}$$

□

Lemma 11.3. *Let $n \ge 64$ be an integer. Let $\epsilon > 0$ and $1/2 > \epsilon_1 > 0$. There exists $\frac{\epsilon}{2n} > \delta > 0$ and a finite subset $\mathcal{G} \subset D \cong M_n$ satisfying the following:*

Let $\Delta, \Delta_1, \Delta_2 : (0, 1) \to (0, 1)$ be increasing maps. Suppose that X is a compact metric space, $\mathcal{F} \subset C(X)$ is a finite subset, $1 > b > a > 0$ and $1 > c > 4\pi/n > 0$. Then there exists a finite subset $\mathcal{F}_1 \subset C(X)$ satisfying the following:

Suppose that A is a unital C^*-algebra with $T(A) \neq \emptyset$, $D \subset A$ is a C^*-subalgebra with $1_D = 1_A$, $\varphi : C(X) \to A$ is a unital contractive completely positive linear map and suppose that $u \in U(A)$ such that

$$\|[x, u]\| < \delta \ \textit{ and } \ \|[x, \varphi(f)]\| < \delta \ \textit{ for all } \ x \in \mathcal{G} \ \textit{ and } \ f \in \mathcal{F}_1. \tag{e 11.559}$$

Suppose also that

$$\tau(\varphi(f)) \ge \Delta(r) \ \textit{ for all } \ \tau \in T(A) \ \textit{ and} \tag{e 11.560}$$

for all $f \in C(X)$ with $0 \le f \le 1$ so that $\{x \in X : f(x) = 1\}$ contains an open ball of X with radius $r \ge a$, and suppose that

$$\tau(\varphi(f)^{1/2}g(u)\varphi(f)^{1/2}) \ge \Delta_1(r)\Delta_2(s) \text{ for all } \tau \in T(A), \tag{e 11.561}$$

for all $f \in C(X)$ with $0 \le f \le 1$ so that $\{x \in X : f(x) = 1\}$ contains an open ball of X with radius $r \ge b$ and for all $g \in C(\mathbb{T})$ with $0 \le g \le 1$ so that $\{t \in \mathbb{T} : g(t) = 1\}$ contains an open arc of $\mathbb{T}$ with length $s \ge c$. Then, there exists a unitary $v \in D$ and a continuous path of unitaries $\{v(t) : t \in [0, 1]\} \subset D$ such that

$$\|[u, v(t)]\| < n\delta < \epsilon, \ \|[\varphi(f), v(t)]\| < n\delta < \epsilon \tag{e 11.562}$$

$$\text{for all } f \in \mathcal{F} \text{ and } t \in [0, 1], \tag{e 11.563}$$

$$v(0) = 1, \ v(1) = v \text{ and} \tag{e 11.564}$$

$$\tau(\varphi(f)^{1/2}g(vu)\varphi(f)^{1/2}) \ge (1 - 1/2^{n+2})\frac{\Delta((1 - 1/2^{n+1})r)}{n^2} \text{ for all } \tau \in T(A) \tag{e 11.565}$$

for any pair of $f \in C(X)$ with $0 \le f \le 1$ so that the set $\{x : f(x) = 1\}$ contains an open ball with radius $r \ge (1 + 1/2^{n+1})a$ and $g \in C(\mathbb{T})$ with $0 \le g \le 1$ so that $\{t \in \mathbb{T} : g(t) = 1\}$ contains an open arc of $\mathbb{T}$ with length $s \ge 4\pi/n + \pi/n2^{n+1}$. Moreover,

$$\tau(\varphi(f)^{1/2}g(vu)\varphi(f)^{1/2}) \ge \Delta_1((1 - 1/2^{n+1})r)\Delta_2((1 - 1/2^{n+1})s) - \frac{\Delta_1(b/2)\Delta_2(c/2)}{2^{n+5}} \tag{e 11.566}$$

for all $\tau \in T(A)$, for all $f \in C(X)$ with $0 \le f \le 1$ so that the set $\{x : f(x) = 1\}$ contains an open ball with radius $r \ge (1 + 2^{-n-1})b$, $i = 0, 1, 2, ..., m_0 - 1$ and $g \in C(\mathbb{T})$ with $0 \le g \le 1$ so that $\{t \in \mathbb{T} : g(t) = 1\}$ contains an open arc of $\mathbb{T}$ with length at least $s \ge (1 + 1/n^2 2^{n+1})c$.

Furthermore,

$$\text{length}(\{v(t)\}) \le \pi. \tag{e 11.567}$$

Proof. Let $r_1 > r_2 > \cdots r_{l-1} > r_l$ and $r_l = b$ such that $r_j - r_{j+1} < a/2^{n+1}$, $j = 1, 2, ..., l - 1$.

Let $0 < \delta_0 = \frac{\epsilon_1\Delta(a/2)\Delta_1(b/2)\Delta_2(c/2)}{128n^2}$. Let $\{e_{i,j}\}$ be a matrix unit for D and let $\mathcal{G} = \{e_{i,j}\}$. Define

$$v = \sum_{j=1}^{n} e^{2\sqrt{-1}j\pi/n}e_{j,j}. \tag{e 11.568}$$

Let $g_j \in C(\mathbb{T})$ with $g_j(t) = 1$ for $|t - e^{2\sqrt{-1}j\pi/n}| < \pi/n$ and $g_j(t) = 0$ if $|t - e^{2\sqrt{-1}j\pi/n}| \ge \pi/n + \pi/n2^{n+2}$ and $1 \ge g_j(t) \ge 0$, $j = 1, 2, ..., n$. As in the proof Lemma 5.1 of [33], we may also assume that

$$g_i(e^{2\sqrt{-1}j\pi/n}t) = g_{i+j}(t) \text{ for all } t \in \mathbb{T} \tag{e 11.569}$$

where $i, j \in \mathbb{Z}/n\mathbb{Z}$. Let $1 = c_0 > c_1 > c_2 > \cdots c_{m_1} = c$ so that $c_j - c_{j+1} < c/n^2 2^{n+1}$, $j = 0, 1, ..., m_1 - 1$.

Define $g_{i,j,c} \in C(\mathbb{T})$ with $0 \le g_{i,j,c} \le 1$, $g_{i,j,c}(t) = 1$ for $|t - e^{2\sqrt{-1}j\pi/n^3 2^{n+1}}| < c_i$ and $g_{i,j,c}(t) = 0$ if $|t - e^{2\sqrt{-1}j\pi/n^3 2^{n+1}}| \ge c_i + c/n^2 2^{n+2}$, $i = 1, 2..., m_1$, $j = 1, 2, ..., n^3 2^{n+1}$. We may also assume that

$$g_{i,j,c}(e^{2\sqrt{-1}k\pi/n}t) = g_{i,j+k',c}(t) \text{ for all } t \in \mathbb{T}, \tag{e 11.570}$$

where $k' = kn^2 2^{n+1}$ and $j, k, k' \in \mathbb{Z}/n^3 2^{n+1}\mathbb{Z}$.

Let $1 = b_0 > b_1 > \cdots b_{m_0} = b$ such that $b_j - b_{j+1} < b/2^{n+1}$, $j = 0, 1, ..., m_0 - 1$. Let $\{x_1, x_2, ..., x_N\}$ be an $a/2^{n+2}$-dense subset of X. Define $f_{i,m} \in C(X)$ with $f_{i,m}(x) = 1$ for $x \in B(x_i, r_m)$ and $f_{i,m}(x) = 0$ if $x \notin B(x_i, r_m + a/2^{n+2})$ and $0 \leq f_{i,m} \leq 1$, $i = 1, 2, ..., N$ and $m = 1, 2, ..., l+1$. Define $f_{i,j,b} \in C(X)$ with $0 \leq f_{i,j,b} \leq 1$, $f_{i,j,b}(x) = 1$ for $x \in B(x_i, b_j)$ and $f_{i,j,b}(x) = 0$ if $x \notin B(x_i, b_j + b/2^{n+1})$, $i = 1, 2, ..., N$, $j = 1, 2, ..., m_0$. Note that

$$\tau(\varphi(f_{i,m})) \geq \Delta(r_m) \text{ for all } \tau \in \mathrm{T}(A), \;\; i = 1, 2, ..., N, j = 1, 2, ..., s+1. \tag{e 11.571}$$

Fix a finite subset $\mathcal{F}_0 \subset C(\mathbb{T})$ which at least contains

$$\{g_1, g_2, ..., g_n\} \cup \{g_{i,j,c} : 1 \leq i \leq m_1, 1 \leq j \leq n\}$$

and $\mathcal{F}_1 \subset C(X)$ which at least contains $\mathcal{F}$, $\{f_{i,m} : 1 \leq i \leq N, 1 \leq m \leq s+1\}$ and $\{f_{i,j,b} : 1 \leq i \leq N, 1 \leq j \leq m_0\}$.

Choose δ so small that the following hold:

(1) there exists a unitary $u_i \in e_{i,i}Ae_{i,i}$ such that $\|e^{2\sqrt{-1}i\pi/n}e_{i,i}ue_{i,i} - u_i\| < \delta_0^2/16n^4 2^{n+6}$, $i = 1, 2, ..., n$;

(2) $\|e_{i,j}g(u) - g(u)e_{i,j}\| < \delta_0^2/16n^4 2^{n+6}$, $\|e_{i,j}\varphi(f) - \varphi(f)e_{i,j}\| < \delta_0^2/16n^4 2^{n+6}$ for $f \in \mathcal{F}_1$ and $g \in \mathcal{F}_0$, i, j, $k = 1, 2, ..., n$;

(3) $\|e_{i,i}g(vu) - e_{i,i}g(e^{2\sqrt{-1}i\pi/n}u)\| < \delta_0^2/16n^4 2^{n+6}$ for all $g \in \mathcal{F}_0$ and

(4) $\|e_{i,j}^* g(u)e_{i,j} - e_{j,j}g(u)e_{j,j}\| < \delta_0^2/16n^4 2^{n+6}$, $\|e_{i,j}^*\varphi(f)e_{i,j} - e_{j,j}\varphi(f)e_{j,j}\| < \delta_0^2/16n^4 2^{n+6}$ for all $f \in \mathcal{F}_1$ and $g \in \mathcal{F}_0$, i, $j = 1, 2, ..., n$.

It follows from (4) that, for any $k_0 \in \{1, 2, ..., N\}$ and $m' \in \{1, 2, ..., l+1\}$,

$$\tau(\varphi(f_{k_0,m'})e_{j,j}) \geq \Delta(r_{m'})/n - n\delta_0^2/16n^4 2^{n+6}. \tag{e 11.572}$$

Fix k_0, m' and k. For each $\tau \in \mathrm{T}(A)$, there is at least one i such that

$$\tau(\varphi(f_{k_0,m'})e_{j,j}g_i(u)) \geq \Delta(r_{m'})/n^2 - \delta_0^2/16n^4 2^{n+6}. \tag{e 11.573}$$

Choose j so that $k + j = i \bmod (n)$. Then, by (e 11.569),

$$\begin{aligned} \tau(\varphi(f_{k_0,m'})g_k(vu)) &\geq \tau(\varphi(f_{k_0,m'})e_{j,j}g_k(e^{2\sqrt{-1}j\pi/n}u))) - \frac{8\delta_0^2}{16n^4 2^{n+6}} && \text{(e 11.574)} \\ &= \tau(\varphi(f_{k_0,m'})e_{j,j}g_i(u)) - \frac{\delta_0^2}{2n^4 2^{n+6}} && \text{(e 11.575)} \\ &\geq \frac{\Delta(r_{m'})}{n^2} - \frac{9\delta_0^2}{16n^4 2^{n+6}} \text{ for all } \tau \in \mathrm{T}(A). && \text{(e 11.576)} \end{aligned}$$

Note again $\tau(xy) = \tau(yx)$ for all $x, y \in A$. It is then easy to compute that

$$\tau(\varphi(f)^{1/2}g(vu)\varphi(f)^{1/2}) \geq \frac{\Delta((1 - 1/2^{n+1})r)}{n^2} - \frac{9\delta_0^2}{16n^4 2^{n+5}} \text{ for all } \tau \in \mathrm{T}(A) \tag{e 11.577}$$

and for any pair of $f \in C(X)$ with $0 \leq f \leq 1$ such that $\{x \in X : f(x) = 1\}$ contains an open ball with radius $r \geq a + a/2^{n+1}$ and $g \in C(\mathbb{T})$ with $0 \leq g \leq 1$ such that $\{t \in \mathbb{T} : g(t) = 1\}$ contains open arc of length at least $4\pi/n + \pi/n^2 2^{n+1}$. One then concludes that

$$\tau(\varphi(f)g(vu)) \geq (1 - 1/2^{n+2})\frac{\Delta((1 - 1/2^{n+1})r)}{n^2} \text{ for all } \tau \in \mathrm{T}(A) \tag{e 11.578}$$

and for any pair of $f \in C(X)$ with $0 \le f \le 1$ such that $\{x \in X : f(x) = 1\}$ contains an open ball with radius $r \ge (1 + 1/2^{n+1})a$ and $g \in C(\mathbb{T})$ with $0 \le g \le 1$ such that $\{t \in \mathbb{T} : g(t) = 1\}$ contains open arc of length at least $s \ge 4\pi/n + \pi/n^2 2^{n+1}$.

On the other hand, by (2), (3) and (4) above,

$$\tau(\varphi(f_{i,j,b})^{1/2} e_{k,k} g_{i',j',c}(vu)\varphi(f_{i,j,b})^{1/2}) \tag{e 11.579}$$

$$\ge \tau(\varphi(f_{i,j,b})^{1/2} e_{k,k} g_{i',j',c}(e^{2\pi\sqrt{-1}k/n}u)\varphi(f_{i,j,b})^{1/2}) - \frac{\delta_0^2}{16n^4 2^{n+6}} \tag{e 11.580}$$

$$= \tau(\varphi(f_{i,j,b})^{1/2} e_{k,k} g_{i',j'+k',c}(u)\varphi(f_{i,j,b})^{1/2}) - \frac{\delta_0^2}{16n^4 2^{n+6}} \tag{e 11.581}$$

$$(k' = kn^2 2^{n+1}) \tag{e 11.582}$$

$$\ge \frac{\Delta_1(b_j)\Delta_2(c_{i'})}{n} - 4n\frac{\delta_0^2}{16n^4 2^{n+6}} \tag{e 11.583}$$

$$\tag{e 11.584}$$

for all $\tau \in T(A)$, $k = 1, 2, ..., n$, $i = 1, 2, ..., N$, $j = 1, 2, ..., m_0$, $i' = 1, 2, ..., m_1$ and $j' = 1, ..., n^2 2^{n+1}$. Thus

$$\tau(\varphi(f_{i,j,b})^{1/2} g_{i',j',a}(vu)\varphi(f_{i,j,b})^{1/2}) \tag{e 11.585}$$

$$\ge \Delta_1(b_j)\Delta_2(a_{j'}) - 4n^2\frac{\delta_0^2}{16n^4 2^{n+6}} \tag{e 11.586}$$

$$= \Delta_1(b_j)\Delta_2(c_{i'}) - \frac{\delta_0^2}{8n^2 2^{n+5}} \tag{e 11.587}$$

for all $\tau \in T(A)$. It then follows

$$\tau(\varphi(f)^{1/2} g(vu)\varphi(f)^{1/2}) \ge \Delta_1(b_i)\Delta_2(c_j) - \frac{\delta_0^2}{n^2 2^{n+5}} \tag{e 11.588}$$

for all $\tau \in T(A)$, for any $f \in C(X)$ with $0 \le f \le 1$ so that $\{x \in X : f(x) = 1\}$ contains an open ball with radius $r \ge (1 + 1/2^{n+1})b_i$ and for any $g \in C(\mathbb{T})$ with $0 \le g \le 1$ so that $\{t \in \mathbb{T} : g(t) = 1\}$ contains an open arc with length $s \ge (1 + 1/n^2 2^{n+1})c_j$, $i = 1, 2, ..., m_0$ and $j = 1, 2, ..., m_1$. From this, one concludes that

$$\tau(\varphi(f)^{1/2} g(vu)\varphi(f)^{1/2}) \ge \Delta_1((1 - 1/2^{n+1})r)\Delta_2((1 - 1/2^{n+1})s) - \frac{\delta_0^2}{n^2 2^{n+5}} \tag{e 11.589}$$

for all $\tau \in T(A)$, for any $f \in C(X)$ with $0 \le f \le 1$ so that $\{x \in X : f(x) = 1\}$ contains an open ball with radius $r \ge (1+1/2^{n+1})b$ and for any $g \in C(\mathbb{T})$ with $0 \le g \le 1$ so that $\{t \in \mathbb{T} : g(t) = 1\}$ contains an open arc with length $s \ge (1 + 1/n^2 2^{n+1})c$.

Note that if $\|[\varphi(f),\, e_{i,i}]\| < \delta$, then

$$\|[\varphi(f), \sum_{i=1}^{n} \lambda_i e_{i,i}]\| < n\delta < \epsilon$$

for any $\lambda_i \in \mathbb{T}$ and $f \in \mathcal{F}_1$. We then also require that $\delta < \epsilon/2n$. Thus, one obtains a continuous path $\{v(t) : t \in [0, 1]\} \subset D$ with $\mathrm{length}(\{v(t)\}) \le \pi$ and with $v(0) = 1$ and $v(1) = v$ so that the second part of (e 11.562) holds.

□

11.4. Let X be a metric space with metric d_0. Define a metric d on $X \times \mathbb{T}$ as follows:

$$d((x, t), (y, r)) = \sqrt{d_0(x, y)^2 + |t - r|^2}$$

for all $x, y \in X$ and $t, r \in \mathbb{T}$.

11.5. Define $\Delta_{00} : (0,1) \to (0,1)$ as follows:

$$\Delta_{00}(r) = \frac{1}{2n^2}, \text{ if } r \in [4\pi/n + \pi/2n^2, 4\pi/(n-1) + \pi/2(n-1)^2) \text{ and } n \ge 64; \quad \text{(e 11.590)}$$

$$\Delta_{00}(r) = \frac{1}{2(64)^2}, \text{ if } r \ge 4\pi/63 + \pi/2(63)^2. \quad \text{(e 11.591)}$$

Define $\Delta_{00}^n : (0,1) \to (0,1)$ as follows.

$$\Delta_{00}^n(r) = \frac{\prod_{j=1}^n (1 - 1/2^{j+1})}{k^2}, \quad \text{(e 11.592)}$$

$$\text{if } r \in [4\pi/k + \sum_{j=k}^n \pi/k^2 2^{j+1}, 4\pi/(k-1) + \sum_{j=k}^n \pi/(k-1)^2 2^{j+1}), \quad \text{(e 11.593)}$$

$$k = 65, ..., n; \quad \text{(e 11.594)}$$

$$\Delta_{00}^n(r) = \frac{\prod_{j=1}^n (1 - 1/2^{j+1})}{(64)^2}, \text{ if } r \ge 4\pi/64 + \pi/2; \quad \text{(e 11.595)}$$

$$\Delta_{00}^n(r) = r\Delta_{00}^n(4\pi/n + \sum_{j=1}^n \pi/k^2 2^{j+1}), \text{ if } r \in (0, 4\pi/n + \sum_{j=k}^n \pi/k^2 2^{j+1}). \quad \text{(e 11.596)}$$

Let $\Delta : (0,1) \to (0,1)$ be an increasing map. Define

$$\Delta_0(\Delta)(r) = \frac{\Delta(r/2)\Delta_{00}(r/2)}{4} \text{ and } \Delta_1(\Delta) = 3\Delta_0(3r/4)/4 \text{ for all } r \in (0,1).$$

Lemma 11.6. *Let X be a compact metric space, let $\epsilon > 0$, let $\mathcal{F} \subset C(X)$ be a finite subset and let $\Delta : (0,1) \to (0,1)$ be an increasing map. Let $\eta \in (0, 1/2)$. Suppose that $\varphi : C(X) \to A$ is a unital contractive completely positive linear map for some unital C^*-algebra A with $T(A) \neq \emptyset$ and $u \in U(A)$ is a unitary such that*

$$\tau(\varphi(g)) \ge \Delta(r) \quad \text{(e 11.597)}$$

for all $g \in C(X)$ with $0 \le g \le 1$ so that $\{x \in X : g(x) = 1\}$ contains an open ball with radius $r \ge \eta/2$. Then there is a unitary $v \in M_K \subset M_K(A)$ and a continuous path of unitaries $\{v_t : t \in [0,1]\} \subset M_K$ such that $v_0 = 1$, $v_1 = v$ and

$$\tau(\varphi^{(K)}(f)g(vu^{(K)})) \ge \Delta(r/2)\Delta_{00}(s/2)/4 \text{ for all } \tau \in T(M_K(A)), \quad \text{(e 11.598)}$$

for all $f \in C(X)$ with $0 \le f \le 1$ so that $\{x \in X : f(x) = 1\}$ contains an open ball of radius $r \ge 4\eta/3$ and for all $g \in C(\mathbb{T})$ with $0 \le g \le 1$ so that $\{t \in \mathbb{T} : g(x) = 1\}$ contains an open arc with length $s \ge 5\eta/2$, where Δ_{00} is defined in 11.5.

Proof. Let ϵ, $\mathcal{F}$, Δ and η be as given. Choose an integer $K_1 \ge 1$ such that $1/K_1 < \eta/16$. Let $K = K_1!/16!$. We will use induction to prove the following:

Suppose that $\varphi : C(X) \to A$ is a unital contractive completely positive linear map for some unital C^*-algebra A with $T(A) \neq \emptyset$ and $u \in U(A)$ is a unitary such that

$$\tau(\varphi(g)) \ge \Delta(r) \quad \text{(e 11.599)}$$

for all $g \in C(X)$ with $0 \le g \le 1$ so that $\{x \in X : g(x) = 1\}$ contains an open ball with radius $r \ge \eta$, Then there is a unitary $v \in M_{n!/32!} \subset M_{n!/32!}(A)$ and a continuous path of unitaries $\{v_t : t \in [0,1]\} \subset M_{n!/32!}$ such that $v_0 = 1$, $v_1 = v$ and

$$\tau(\varphi^{(n!/32!)}(f)g(vu^{(n!/32!)})) \ge \prod_{k=32}^n (1 - 1/2^{k+2})\Delta((\prod_{k=32}^n (1 - 1/2^{k+1})r))\Delta_{00}^n(r) \quad \text{(e 11.600)}$$

for all $\tau \in T(M_{n!/32!}(A))$, for all $f \in C(X)$ with $0 \le f \le 1$ so that $\{x \in X : f(x) = 1\}$ contains an open ball of radius $r \ge \prod_{k=32}^n (1 + 1/2^{k+2})\eta$ and for all $g \in C(\mathbb{T})$ with $0 \le g \le 1$ so that $\{t \in \mathbb{T} : g(x) = 1\}$ contains an open arc with length $s \ge 4\pi/(n-1) + \sum_{k=32!}^n \pi/k^2 2^{k+1}$, where Δ_0 is defined in 11.5. We use induction on n.

Let $n = 64$. Consider $\varphi^{(64)}$ and $u^{(64)}$ and $D = M_{64} \subset M_{64}(A)$. Note that $x^{(64)}d = dx^{(64)}$ for all $x \in A$ and $d \in D$. We choose $a = \eta/2$ and ignore b, c and the last part of statement of 11.3 after "Moreover." We also use $\delta = 0$. Lemma 4.4 implies that the above statement holds for $n = 64$. Denote by $v_{64} \in M_{64}$ provided by 11.3 ($n = 64$).

We now assume that the above statement holds for some $n \ge 64$. Denote by v_n the unitary $v \in M_{n!/32!}$ provided by the above statement for n. Let $D = M_{n+1}$. We write $M_{(n+1)!/32!} = M_{n!/32!} \otimes D$ and consider $A \otimes M_{n!/64} \otimes D$ instead of A. Put $\Delta_1 = (\prod_{k=64}^n (1 - 1/2^{k+2}))\Delta((\prod_{k=64}^n (1 - 1/2^{k+1})r)$ and $\Delta_2(s) = \Delta_{00}^n(s)$. Choose $a = \eta/2$, $b = \eta/2$ and $c = 2\pi/n$. Consider $\varphi^{(n!/32!)}$ and

$$U_n = v_n v_{n-1}^{(n)} \cdots v_{64}^{(n!/64!)} u^{(n!/32!)}$$

(in place of u). We then applying Lemma 11.3 again. It follows that there is a unitary $v_{n+1} \in D = M_{n+1}$ and a continuous path of unitaries $\{v_{n+1}(t) : t \in [0,1]\} \subset M_{n+1}$ such that $v_{n+1}(0) = v_{n+1}$, $v_{n+1}(1) = 1$,

$$\tau(\varphi^{(n+1)!/32!)}(f)g(v_{n+1}U_n^{(n+1)})) \ge (1 - 2^{n+3})\Delta((1 - 1/2^{n+2})r)/(n+1)^2 \qquad \text{(e 11.601)}$$

for all $\tau \in T(M_{(n+1)!/32!}(A))$, for all $f \in C(X)$ with $0 \le f \le 1$ so that $\{x \in X; f(x) = 1\}$ contains an open ball of radius $r \ge (1 - 2^{n+1})\eta/2$ and for all $g \in C(\mathbb{T})$ with $0 \le g \le 1$ so that $\{t \in \mathbb{T} : g(x) = 1\}$ contains an open arc of length $s \ge 4\pi/(n+1) + \pi/(n+1)2^{n+1}$, and

$$\begin{aligned} &\tau(\varphi^{(n+1)!/32!)}(f)g(v_{n+1}U_n^{(n+1)})) && \text{(e 11.602)} \\ &\ge \Delta_1((1 - 2^{n+2})r)\Delta_2((1 - 1/2^{n+2})s) - \frac{\Delta_1(\eta/2)\Delta_{00}^n(\pi/n))}{2^{n+6}} && \text{(e 11.603)} \\ &\ge \prod_{k=64}^{n+1} (1 - 1/2^{k+1})\Delta((\prod_{k=64}^{n} (1 - 1/2^{k+1})r)\Delta_{00}^{n+1}(s) && \text{(e 11.604)} \end{aligned}$$

for all $\tau \in T(M_{(n+1)!/32!}(A))$, for all $f \in C(X)$ with $0 \le f \le 1$ so that $\{x \in X; f(x) = 1\}$ contains an open ball of radius $r \ge (1 - 2^{n+1})\eta/2$ and for all $g \in C(\mathbb{T})$ with $0 \le g \le 1$ so that $\{t \in \mathbb{T} : g(x) = 1\}$ contains an open arc of length $s \ge 4\pi/n + \pi/(n+1)2^{n+1}$.

This proves the above statement for $n+1$ and ends the induction. It follows that the lemma follows.

□

Lemma 11.7. *Let X be a compact metric space, let $\epsilon > 0$, let $\mathcal{F} \subset C(X)$ be a finite subset and let $\Delta : (0,1) \to (0,1)$ be an increasing map. Let $\eta \in (0, 1/2)$. There exists $\delta > 0$, a finite subset $\mathcal{G} \subset C(X)$ and an integer $K \ge 1$ satisfying the following: Suppose that $\varphi : C(X) \to A$ is a unital contractive completely positive linear map for some unital C^*-algebra A with $T(A) \neq \emptyset$ and $u \in U(A)$ is a unitary such that*

$$\|[\varphi(g), u]\| < \delta \textit{ for all } g \in \mathcal{G} \text{ and } \tau(\varphi(g)) \ge \Delta(r) \qquad \text{(e 11.605)}$$

for all $g \in C(X)$ with $0 \le g \le 1$ so that $\{x \in X : g(x) = 1\}$ contains an open ball with radius $r \ge \eta/2$, Then there is a continuous path of unitaries $\{u_t : t \in [0,1]\} \subset M_K(A)$ such that $u_0 = u^{(K)}$ and $u_1 = U$ and

$$\|[\varphi^{(K)}(f), u_t]\| < \epsilon \textit{ for all } f \in \mathcal{F}, \qquad \text{(e 11.606)}$$

and there exists a unital contractive completely positive linear map $\Phi: C(X)\times C(\mathbb{T})\to M_K(A)$ *such that*

$$\|\Phi(f\otimes 1)-\varphi^{(K)}(f)\|<\epsilon \text{ for all } f\in\mathcal{F},\ \ \|\Phi(1\otimes z)-U\|<\epsilon \text{ and} \tag{e 11.607}$$
$$\mu_{\tau\circ L}(O_r)\ge \Delta_1(\Delta)(r) \tag{e 11.608}$$

for all $\tau\in T(M_K(A))$ *and for open balls of* $X\otimes\mathbb{T}$ *of radius* $r\ge 5/2\eta$.

Proof. Let $\epsilon>0$, $\mathcal{F}\subset C(X)$ be a finite subset, $\Delta:(0,1)\to(0,1)$ be an increasing map and let $\eta\in(0,1/2)$. To simplify notation, without loss of generality, we may assume that $\mathcal{F}$ is a subset of the unit ball of $C(X)$.

Let $0<\delta_0<\min\{\epsilon/2, \Delta_0(\Delta)(\eta/8)/16\}$. Let $\mathcal{F}_1$ be a finite subset. There exists $\epsilon/2>\delta>0$ and a finite subset $\mathcal{G}\subset C(X)$ containing $\mathcal{F}$ satisfying the following: For any unital contractive completely positive linear map $\psi: C(X)\to C$ (for any unital C^*-algebra) and any unitary $W\in C$ with

$$\|[\psi(g), W]\|<\delta \text{ for all } g\in\mathcal{G}_1,$$

there exists a unital contractive completely positive linear map $L: C(X)\otimes C(\mathbb{T})\to C$ such that

$$\|L(f\otimes 1)-\psi(f)\|<\delta_0/2 \text{ for all } f\in\mathcal{F}_1 \text{ and } \|L(1\otimes z)-W\|<\delta_0/2.$$

Let $K_1\ge 1$ be an integer such that $1/K_1<\eta/16$. Let $K=K_1!/32!$. Suppose that φ and u satisfy the assumption of the lemma for the above δ and $\mathcal{G}$. By applying 11.6, we obtain a unitary $v\in M_K\subset M_K(A)$ and a continuous path of unitaries $\{u_t: t\in[0,1]\}\subset M_K$ such that $u_0=1$, $u_1=v$ and

$$\tau(\varphi^{(K)}(f)g(vu^{(K)}))\ge \Delta(r/2)\Delta_{00}(s/2)/4 \text{ for all } \tau\in T(M_K(A)), \tag{e 11.609}$$

for all $f\in C(X)$ with $0\le f\le 1$ so that $\{x\in X: f(x)=1\}$ contains an open ball of radius $r\ge 4\eta/3$ and for all $g\in C(\mathbb{T})$ with $0\le g\le 1$ so that $\{t\in\mathbb{T}: g(t)=1\}$ contains an open arc with length $s\ge 5\eta/2$.

Note that

$$\|[\varphi^{(K)}(g),\, vu^{(K)}]\|<\delta<\epsilon \text{ for all } g\in\mathcal{G}. \tag{e 11.610}$$

It follows that there exists a unital contractive completely positive linear map $\Phi: C(X)\otimes C(\mathbb{T})\to M_K(A)$ such that

$$\|\Phi(f\otimes 1)-\varphi^{(K)}(f)\| < \delta_0/2<\epsilon \text{ for all } f\in\mathcal{F}_1 \text{ and} \tag{e 11.611}$$
$$\|\Phi(1\otimes z)-vu^{(K)}\| < \delta_0/2<\epsilon. \tag{e 11.612}$$

Define $U=vu^{(K)}$. With sufficiently large $\mathcal{F}_1$ which can be determined before choosing δ and $\mathcal{G}$, we have

$$\mu_{\tau\circ L}(O_r)\ge\Delta_1(\Delta)(r) \text{ for all } \tau\in T(M_K(A)) \tag{e 11.613}$$

and for all open ball O_r with radius $r\ge 2\eta$. ☐

The proof of 11.1 under the assumption that 10.4 (and 10.5) hold for finite CW complexes Y with $\dim Y\le d$

Let $\epsilon>0$, $\mathcal{F}\subset C(X)$ and Δ be given as in 11.1. Define $\Delta':(0,1)\to(0,1)$ by $\Delta'(r)=\Delta(15r/16)$. Let $\Delta_1(\Delta')$ be as defined in 11.5. Let η, δ_1 (in place of δ), $\mathcal{G}_1$ (in place of $\mathcal{G}$) $\mathcal{P}$, $\mathcal{Q}$, $N\ge 1$ and K_1 (in place of K) be as required by 11.2 for ϵ, $\mathcal{F}$ and $\Delta_1(\Delta')$.

Let $\delta_2 > 0$ (in place of δ), $\mathcal{G}_2 \subset C(X)$ (in place of $\mathcal{G}$) be a finite subset and let $K_2 \geq 1$ (in place of K) be an integer required by 11.7 for $\min\{\epsilon/2, \delta_1\}$ (in place of ϵ), $\mathcal{G}_1 \cup \mathcal{F}$ (in place of $\mathcal{F}$), $\Delta_1(\Delta')$ (in place of Δ) and $\eta/16$ (in place of η).

Let $\delta = \min\{\delta_2, \delta_1/2, \epsilon/4\}$ and let $\mathcal{G} = \mathcal{F} \cup \mathcal{G}_1 \cup \mathcal{G}_2$. Put $K = K_1 K_2!$.

Now let Y be a finite CW complex with $\dim Y \leq d$ and $C = PM_m(C(Y))P$ for some projection $P \in M_m(C(Y))$ with $\mathrm{rank} P(y) \geq N \geq 1$, let $\varphi : C(X) \to C$ be a unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear map and let $u \in U(C)$ be a unitary which satisfy the assumption of 11.1 for the above η, γ, δ, $\mathcal{G}$, $\mathcal{P}$ and $\mathcal{Q}$. It follows from 11.7 that there is a continuous path of unitaries $\{w_t : t \in [0, 1/2]\} \subset M_{K_2!}(C)$ such that $w_0 = u^{(K_2!)}$,

$$\|[\varphi^{(K_2!)}(g),\, w_t]\| < \min\{\epsilon/2, \delta_1\} \text{ for all } t \in [0, 1/2], \;\; g \in \mathcal{G}_1 \cup \mathcal{F} \qquad \text{(e 11.614)}$$

and there exists a unital contractive completely positive linear map $\Phi : C(X) \otimes C(\mathbb{T}) \to M_{K_2!}(C)$ such that

$$\|\Phi(f \otimes 1) - \varphi^{(K_2!)}(f)\| < \min\{\epsilon/2, \delta_1\} \text{ for all } f \in \mathcal{G}_1 \cup \mathcal{F}, \qquad \text{(e 11.615)}$$

$$\|\Phi(1 \otimes z) - w_{1/2}\| < \min\{\epsilon/2, \delta_1\} \text{ and} \qquad \text{(e 11.616)}$$

$$\mu_{\tau \circ \Phi}(O_r) \geq \Delta_1(\Delta')(r) \text{ for all } \tau \in M_{K_2!}(C) \qquad \text{(e 11.617)}$$

and for all open balls O_r with radius $r \geq 5\eta/32$.

Then, by 11.2, there is a continuous path of unitaries $\{v_t : t \in [1/2, 1]\} \subset M_{K_1 K_2!}(C)$ with $v_{1/2} = w_1$ and $v_1 = 1$ such that

$$\|[\varphi^{(K)}(f),\, v_t]\| < \epsilon \text{ for all } t \in [1/2, 1], \quad \text{and} \quad \text{for all } f \in \mathcal{F}. \qquad \text{(e 11.618)}$$

Now define

$$u_t = \begin{cases} w_t^{(K_1)} & \text{if } t \in [0, 1/2]; \\ v_t, & \text{if } t \in (1/2, 1]. \end{cases}$$

Note that $u_0 = u^{(K)} = \mathrm{diag}(\overbrace{u, u, ..., u}^{K})$. This path meets the requirements.

12 The proof of the uniqueness theorem 10.4

Proof of 10.4

The case that Y is a single point is well known. A reference can be found in Theorem 2.10 of [36]. The case that Y is a set of finitely many points follows. The case that $Y = [0, 1]$ has been proved in Theorem 3.6 of [36].

We now assume that 10.4 holds for the case that Y is any finite CW complex of dimension no more than $d \geq 0$. We will use it to show that 10.4 holds for the case that Y is any finite CW complex of dimension no more than $d + 1$. Then induction implies that 10.4 holds for any integer $d \geq 0$. Note now 11.1 and 10.6 hold for Y being a finite CW complex with $\dim Y \leq d$.

Let $\epsilon > 0$ and let $\mathcal{F} \subset C(X)$ be a finite subset. To simplify notation, without loss of generality, we may assume that $\mathcal{F}$ is in the unit ball of $C(X)$. Let $\eta_1' > 0$ (in place of η), $\delta_1 > 0$ (in place of δ), $\gamma_0 > 0$ (in place of γ), $\mathcal{G}_1 \subset C(X)$ (in place of $\mathcal{G}$) be a finite subset, let $\mathcal{P}_0 \subset \underline{K}(C(X))$ (in place of $\mathcal{P}$) be a finite subset and $\mathcal{Q} \subset \ker_{\rho_{C(X)}}$ be a finite subset, let N_1 (in place of N) and an integer K_1 (in place K) required by 11.1 for $\epsilon/32$ (in place of ϵ), $\mathcal{F}$ and $\Delta(r/3)/3$ (in place of Δ). We may assume that $\mathcal{Q} \subset K_0(C(X)) \cap \mathcal{P}_0$. Let $\eta_1 = \eta_1'/3$. We may assume that $\delta_1 < \epsilon/32K_1$.

Write $C(X) = \lim_{n\to\infty}(C(X_n), \imath_n)$, where each X_n is a finite CW complex and $\imath_n$ is a unital homomorphism. Let $K_2 \geq 1$ be an integer such that $K_2 x = 0$ for any $x \in \mathrm{Tor}(K_i(C(X))) \cap \mathcal{P}_0$

$(i=0,1)$. Let $K_3=K_1\cdot K_2!$. We may also assume that, for any δ_2-$\{z,1\}\times\mathcal{G}_2$-multiplicative contractive completely positive linear map $\Lambda: C(\mathbb{T})\otimes C(X)\to C$ (for any unital C^*-algebra C with $T(C)\neq\emptyset$), $[\Lambda]$ is well defined and

$$\tau([\Lambda((g))])=0$$

for all $g\in Tor(K_1(C(X)))\cap\mathcal{P}_0$. Furthermore, we may assume that δ_2 is so small and $\mathcal{G}_2$ is so large that $\mathrm{Bott}(\psi',v)|_{\mathcal{P}_0}$ is well defined for any unital homomorphism ψ' from $C(X)$ and unitary v in the target algebra such that $\|[\varphi'(g),\,v]\|<3\delta_2$ for all $g\in\mathcal{G}_2$. Moreover if $\|v_1-v_2\|<3\delta_2$, then

$$\mathrm{Bott}(\varphi',v_1)|_{\mathcal{P}_0}=\mathrm{Bott}_1(\varphi',v_2)|_{\mathcal{P}_0}.$$

We also assume that, if there are unitaries u_1,u_2,v_1,v_2,v_3,v_4 and projections e_1,e_2 such that

$$\|u_1-u_2\|<3\delta_2,\ \ \|e_1-e_2\|<3\delta_2\ \ \|[e_i,v_j]\|<3\delta_2\ \text{ and }\ \|[u_i,v_j]\|<3\delta_2, \tag{e 12.619}$$

$i=1,2$ and $j=1,2,3,4$, then

$$\mathrm{bott}_0(e_1,v_j)=\mathrm{bott}_0(e_2,v_j),\ \ \mathrm{bott}_1(u_1,v_j)=\mathrm{bott}_1(u_2,v_j), \tag{e 12.620}$$

$$\mathrm{bott}_1(e_1,v_1v_2v_3v_4)=\sum_{j=1}^4\mathrm{bott}_0(e_1,v_j)\ \text{ and} \tag{e 12.621}$$

$$\mathrm{bott}_1(u_1,v_1v_2v_3v_4)=\sum_{j=1}^4\mathrm{bott}_1(u_1,v_j). \tag{e 12.622}$$

We assume that $m(X)\ge 1$ is an integer and $g_i\in U(M_{m(X)}(C(X)))$ so that $\{[g_1],[g_2],...,[g_{k(X)}]\}$ forms a set of generators for $K_1(C(X))\cap\mathcal{P}_0$. We also assume that $[g_j]\neq 0$, $j=1,2,...,k(X)$. Let $\mathcal{U}=\{g_i,g_2,...,g_{k(X)}\}$. We may also assume that $m(X)\ge R(G(\mathcal{U}))$.

Let $\mathcal{S}_1\subset C(X)$ be a finite subset such that

$$\mathcal{U}=\{(a_{i,j})\in\mathcal{U}: a_{i,j}\in\mathcal{S}_1\}.$$

We may assume that $\mathcal{P}_1=\{p_1,p_2,...,p_{k_0(X)}\}\subset M_{m(X)}(C(X))$ is a finite subset of projections such that $\mathcal{P}_1=K_0(C(X))\cap\mathcal{P}_0$. Let $\mathcal{S}_0\subset C(X)$ be a finite subset such that

$$\mathcal{P}_1=\{(b_{ij}): b_{i,j}\in\mathcal{S}_0\}.$$

Moreover, we may assume, without loss of generality, that $\mathcal{Q}\subset\{[p_i]-[p_j]: 1\le i,j\le k_0\}$. We may also assume that $m(X)\ge R(G(\mathcal{P}_1))$.

Let

$$\delta_u'=(\frac{1}{K_3(d+1+m(X))^2})\min\{1/256,\delta_1/16,\delta_2/16,\gamma_0/16\}$$

and $\mathcal{G}_u'=\mathcal{F}\cup\mathcal{G}_1\cup\mathcal{G}_2\cup\mathcal{S}_0\cup\mathcal{S}_1$. Let $\eta_2>0$ (in place of η), $\delta'>0$ (in place of δ), $\mathcal{G}'\subset C(X)$ (in place of $\mathcal{G}$) be a finite subset, $n_0\ge 1$, $N_2\ge 1$ (in place of N) be an integer, $K_4\ge 1$ (in place of K) be an integer required by 10.6 for $\delta_u'/2$ (in place of ϵ), $\gamma_0/(d+1+m(X))$ (in place of δ_0), $\mathcal{G}_u'$ (in place of $\mathcal{G}$), $\mathcal{P}_0$ (in place of $\mathcal{P}$) and $\mathcal{Q}$. We may assume that $\mathcal{P}_0\subset[\psi_{n_0,\infty}](\underline{K}(C(X_{n_0})))$. Furthermore, we may assume, without loss generality, that there are unitaries $g_i'\in M_{m(X)}(C(X_{n_0}))$ such that $\psi_{n_0,\infty}\otimes\mathrm{id}_{M_{m(X)}}(g_i')=g_i$, $i=1,2,...,k(X)$, and there are projections $p_j'\in M_{m(X)}(C(X_{n_0}))$ such that $\psi_{n_0,\infty}\otimes\mathrm{id}_{M_m(X)}(C(X_{n_0}))$, $j=1,2,...,k_0$. Without loss of generality, we may assume that $K_3|K_4$. We may also assume that $K_4x=0$ for all $x\in\mathrm{Tor}(K_i(C(X_{n_0}))$, $i=0,1$.

To simplify notation, without loss of generality, by adding more projections, we may further assume that $\{p_1',p_2',...,p_{k_0}'\}$ generates $K_0(C(X_{n_0})$, and by adding more unitaries, we may assume that $\{g_1',g_2',...,g_{k(X)}'\}$ generates $K_1(C(X_{n_0}))$.

Let $\delta_u = \min\{\delta'_u, \delta'/2\}$ and $\mathcal{G}_u = \mathcal{G}'_u \cup \mathcal{G}'$. Let $\delta_3 > 0$ (in place of δ) and let $\mathcal{G}'_3 \subset C(\mathbb{T}) \otimes C(\mathbb{T})$ (in place of $\mathcal{G}$) be required by Lemma 10.3 of [34] for $1/4NK_4m(X)$ (in place of σ) and $\mathbb{T} \times \mathbb{T}$ (in place of X). Without loss of generality, we may assume that $\mathcal{G}'_3 = \{1 \otimes 1, 1 \otimes z, z \otimes 1\}$. Let

$$\mathcal{G}_3 = \{z \otimes g : g \in \mathcal{G}_u\} \cup \{1 \otimes g : g \in \mathcal{G}_u\}.$$

Let $\epsilon'_1 > 0$ (in place of δ) and let $\mathcal{G}_4 \subset C(X)$ (in place of $\mathcal{G}$) be a finite subset required by 3.4 of [36] for $\min\{\eta_1/2, \eta_2/2\}$, $3/4$ (in place of λ_1) and $1/4$ (in place of λ_2).

Let $\epsilon''_1 = \min\{1/27K_3K_4(d+1+m(X))^2, \delta_u/K_3K_4(2d+2+m(X))^2, \delta_3/2K_3K_4(d+1+m(X))^4, \epsilon'_1/2K_3K_4(d+1+m(X))^2, \gamma_0/16K_3K_4(d+1+m(X))^2\}$ and let $\bar{\epsilon}_1 > 0$ (in place of δ) and $\mathcal{G}_5 \subset C(X)$ (in place of $\mathcal{F}_1$) be a finite subset required by 2.8 of [31] for ϵ''_1 (in place of ϵ) and $\mathcal{G}_u \cup \mathcal{G}_4$ (and $C(X)$ in place of B). Put

$$\epsilon_1 = \min\{\epsilon'_1, \epsilon''_1, \bar{\epsilon}_1\}.$$

Let $\eta_3 > 0$ (in place of η), $1/4 > \gamma_1 > 0$, $1/4 > \gamma'_2 > 0$ (in place of γ_2), $\delta_4 > 0$(in place of δ), $\mathcal{G}_6 \subset C(X)$ (in place of $\mathcal{G}$), $\mathcal{H} \subset C(X)$ be a finite subset, let $\mathcal{P}_2 \subset \underline{K}(C(X))$ (in place of $\mathcal{P}$), let $N_3 \geq 1$ (in place of N) and let $K_5 \geq 1$(in place of K) be required by 10.4 for $\epsilon_1/2^8(m(X)+d+1)^2$ (in place of ϵ), $\mathcal{G}_u \cup \mathcal{G}_4 \cup \mathcal{G}_5$ (in place of $\mathcal{F}$), Δ and for dimY $\leq d$. Let $\eta = \min\{\eta_1/4, \eta_2/4, \eta_3/4\}$. Let $\delta = \min\{\epsilon_1/4, \delta_u/4, \delta_3/4m(X)^2, \delta_4/4, \delta_5/4\}$, $\mathcal{G} = \mathcal{G}_u \cup \mathcal{G}_4 \cup \mathcal{G}_5 \cup \mathcal{G}_6 \cup \mathcal{G}_7 \cup \mathcal{H}$ and $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_2$. Let $\gamma_2 < \min\{\gamma'_2/16(d+1+m(X))^2, \delta_u/9(d+1+m(X))^2, 1/256N_1(d+1+m(X))^2\}$. We may assume that $(\delta, \mathcal{G}, \mathcal{P})$ is a KL-triple. Denote $\eta = \min\{\eta_3, \eta_4\}$. Let $N = 4(k(X) + k_0(X) + 1)\max\{N_1, N_2, N_3\}$. We also assume that η is smaller than the one required by 10.3 for $\epsilon_1/4$ and $\mathcal{U}$. We also assume that γ_1, γ_2 and δ are smaller, and $\mathcal{H}$, $\mathcal{G}$, $\mathcal{P}$, $\mathcal{V}$ and N are larger than required by 10.3 for $\epsilon_1/4$ and $\mathcal{U}$ as well as Δ.

Now suppose that $\varphi, \psi : C(X) \to C = PM_r(C(Y))P$ are unital δ-$\mathcal{G}$-multiplicative contractive completely positive linear maps, where Y is a finite CW complex of dimension $d+1$ and $\mathrm{rank}P(y) \geq N$ for all $y \in Y$, which satisfy the assumption for the above η, δ, γ_i $(i = 1, 2)$, $\mathcal{P}$, $\mathcal{V} = K_1(C(X) \cap \mathcal{P}$ and $\mathcal{H}$. To simplify notation, without loss of generality, we may write φ and ψ instead of $\varphi^{(K_5)}$ and $\psi^{(K_5)}$, respectively, and we also write C instead of $M_{K_5}(C)$.

Let $\bar{p}_{j,(1)}$, $\bar{p}_{j,(2)} \in C \otimes M_{m(X)}$ be a projection such that

$$\|\bar{p}_{j,(1)} - \varphi \otimes \mathrm{id}_{M_{m(X)}}(p_j)\| < \epsilon_1/16 \text{ and} \tag{e 12.623}$$

$$\|\bar{p}_{j,(2)} - \psi \otimes \mathrm{id}_{M_{m(X)}}(p_j))\| < \epsilon_1/16 \;\; j = 1, 2, ..., k_0(X). \tag{e 12.624}$$

Let $\bar{g}_{j,(1)}, \bar{g}_{j,(2)} \in C \otimes M_{m(X)}$ be a unitary such that

$$\|\bar{g}_{j,(1)} - \varphi \otimes \mathrm{id}_{M_{m(X)}}(g_j)\| < \epsilon_1/16 \text{ and} \tag{e 12.625}$$

$$\|\bar{g}_{j,(2)} - \psi \otimes \mathrm{id}_{M_{m(X)}}(g_j)\| < \epsilon_1/16, \;\; j = 1, 2, ..., k(X). \tag{e 12.626}$$

Since $\varphi_{*0}([p_i]) = \psi_{*0}([p_i])$, there is a unitary $X_{0,i} \in M_{m(X)+d+1}(C)$ such that

$$X_{0,i}(\bar{p}_{i,(1)} \oplus \mathrm{id}_C^{(d+1)})X_{0,i}^* = \bar{p}_i^{(2)} \oplus \mathrm{id}_C^{(d+1)}, \;\; i = 1, 2, ..., k_0(X). \tag{e 12.627}$$

It follows from 10.3 that there is a unitary $X_{1,j} \in M_{m(X)}(C)$ such that

$$\|X_{1,j}\bar{g}_{j,(1)}X_{1,j}^* - \bar{g}_j^{(2)}\| < \epsilon_1/4, \tag{e 12.628}$$

$j = 1, 2, ..., k(X)$.

To simplify notation, without loss of generality, we may assume that Y is connected. Let $n = \mathrm{rank}\, P$. Let $Y^{(d)}$ be the d-skeleton of Y. There is a compact subset Y'_d of Y which contains $Y^{(d)}$ and which is a d-dimensional finite CW complex and satisfies the following:

(1) $Y \setminus Y_d'$ is a finitely many disjoint union of open $d+1$-cells: $D_1, D_2, ..., D_R$;

(2) $\|X_{i,j}(y) - X_{i,j}(\xi)\| < \epsilon_1/16$, $i = 1, 2$ and $j = 1, 2, ..., k_0(X)$,

(3) $\|\bar{p}_{j,(i)}(y)) - \bar{p}_{j,(i)}(\xi)\| < \epsilon_1/16$, $j = 1, 2, ..., k_0(X)$, $i = 1, 2$,

(4) $\|\bar{g}_{j,(i)}(y)) - \bar{g}_{j,(i)}(\xi)\| < \epsilon_1/16$, $j = 1, 2, ..., k_0(X)$, $i = 1, 2$,

(5) $\|\varphi(g)(y) - \varphi(g)(\xi)\| < \epsilon_1/16$ and

(6) $\|\psi(g)(y) - \psi(g)(\xi)\| < \epsilon_1/16$ for all $g \in \mathcal{G}$, where ξ is in one of the $d+1$-cells and y is in the boundary (in Y) of the $d+1$-cell.

Denote by $\xi_j \in D_j$ the center of D_j, where we view D_j as an open $d+1$-ball. Let $Y_d = Y_d' \sqcup \{\xi_1, \xi_2, ..., \xi_R\}$. Let $B = \pi(C)$, where $\pi(f) = f|_{Y_d}$. We may write $B = P'M_r(C(Y_d))P'$, where $P' = P|_{Y_d}$.

By applying 10.4 for finite CW complex with dimension no more than d, for each i, there exists a unitary $w \in B$ such that

$$\|w\varphi(g)w^* - \psi(g)\|_{Y_d} < \epsilon_1/2^8(m(X)+d+1)^2 \text{ for all } g \in \mathcal{G}_5. \tag{e 12.629}$$

Recall, to simplify notation, that we write φ and ψ instead of $\varphi^{(K_5)}$ and $\psi^{(K_5)}$, respectively, and we also write C instead of $M_{K_5}(C)$. Note that we have that

$$\mu_{\tau\circ\varphi}(O_r) \ge \Delta(r) \text{ for all } r \ge \eta \text{ and } \text{ for all } \tau \in T(B). \tag{e 12.630}$$

Let $W = w \otimes \mathrm{id}_{M_{m(X)}}$. Then

$$\|[\bar{g}_{j,(1)}, X_{1,j}|_{Y_d}^* W]\| < \epsilon_1/4, \ \ j = 1, 2, ..., k(X). \tag{e 12.631}$$

So $\mathrm{bott}_1(\varphi \otimes \mathrm{id}_{M_{m(X)}}(g_j), X_{1,j}|_{Y_d}^* W)$ is well defined. Let $\alpha_1 : K_1(C(X_{n_0})) \to K_0(B)$ be defined by $\alpha_1(g_j') = \mathrm{bott}_1(\varphi \otimes \mathrm{id}_{M_{m(X)}}(g_j), X_{1,j}|_{Y_d}^* W)$, $j = 1, 2, ..., k(X)$. Let $B' = B|_{Y_d'}$ and let $\pi' : B \to B'$ be the quotient map induced by the restriction. Let Y_i be the boundary of D_i in Y_d, $i = 1, 2, ..., R$. Let $B_i = C|_{Y_i}$ and $\pi_i : B \to B_i$ be the surjective map induced by the restriction, $i = 1, 2, ..., R$. Let $\alpha_{1,i} : K_1(C(X_{n_0})) \to K_0(B_i)$ by $\alpha_{1,i} = (\pi_i)_{*1} \circ \alpha_1$. Let $B_i' = C|_{\{\xi_i\}}$ and $\pi_i' : B \to B_i'$ be the quotient map, $i = 1, 2, ..., R$. Define $\alpha_{1,i}' = (\pi_i')_{*1} \circ \alpha_1$. Note that

$$\alpha_1'(g_j') = \mathrm{bott}_1((\varphi \otimes \mathrm{id}_{M_{m(X)}})|_{Y_d'}(g_j), X_{1,j}^*|_{Y_d'} W|_{Y_d'}), \tag{e 12.632}$$
$$\alpha_{1,i}(g_j') = \mathrm{bott}_1((\varphi \otimes \mathrm{id}_{M_{m(X)}})(g_j')|_{Y_i}, X_{1,j}^*|_{Y_i} W|_{Y_i}) \text{ and} \tag{e 12.633}$$
$$\alpha_{1,i}'(g_j') = \mathrm{bott}_1((\varphi \otimes \mathrm{id}_{M_{m(X)}})(g_j)(\xi_i), X_{1,j}^*(\xi_i) W(\xi_i)) \tag{e 12.634}$$

$j = 1, 2, ..., k(X)$ and $i = 1, 2, ..., R$. Note that, by (e 12.631), 10.3 of [34] (in the connection of 2.8 of [31]) and by the choice of δ_3, ϵ_1'' and $\mathcal{G}_u'$, we have

$$|\rho_B(\alpha_1(g_j))(\tau)| < 1/4N_2K_4m(X), \quad |\rho_{B'}(\alpha_1'(g_j))(\tau)| < 1/4N_2K_4m(X), \tag{e 12.635}$$
$$|\rho_{B_i}(\alpha_{1,i}(g_j))(\tau)| < 1/4N_2K_4m(X) \text{ and } |\rho_{B_i'}(\alpha_{1,i}'([g_j]))(\tau)| < 1/4N_2K_4m(X), \tag{e 12.636}$$

$j = 1, 2, ..., k(X)$ and $i = 1, 2, ..., R$, and for all $\tau \in T(B)$, $\tau \in T(B')$, $\tau \in T(B_i)$ and $\tau \in T(B_i')$, respectively.

Denote by $q_{0,j} = \bar{p}_{j,(1)} \oplus \mathrm{id}_C^{(d+1)}$ and $q_{0,j}' = \bar{p}_{j,(2)} \oplus \mathrm{id}_C^{(d+1)}$, $j = 1, 2, ..., k_0(X)$. Let $\bar{W} = w \otimes \mathrm{id}_{M_{m(X)+d+1}} = w^{(d+1+m(X))} \in M_{m(X)+d+1}(B)$. Then we have

$$\|\bar{W} q_{0,i}|_{Y_d} \bar{W}^* - q_{0,j}'|_{Y_d}\| < (m(X)+d)^2\epsilon_1/4 < \min\{\delta_u/8, \gamma_0/64\}. \tag{e 12.637}$$

There is a unitary $\Theta_i \in M_{m(X)+d}(B)$ such that

$$\|\Theta_i - 1\| < \delta_u/4 \text{ and } (\Theta_i \bar{W})q_{0,i}|_{Y_d}(\Theta_i \bar{W})^* = q_{0,i}'|_{Y_d}, \tag{e 12.638}$$

$i = 1, 2, ..., k_0(X)$.

Define

$$z_j = (\mathrm{id}_{M_{m(X)+d+1}(B)} - q_{0,1}|_{Y_d}) \oplus X_{0,j}^*|_{Y_d}(\Theta_i \bar{W})q_{0,1}|_{Y_d}, \ \ j = 1, 2, ..., k_0(X).$$

It follows that

$$\mathrm{bott}_0(\varphi, \, X_{0,j}^*|_{Y_d}\bar{W})([p_j]) = [z_j], \tag{e 12.639}$$

$j = 1, 2, ..., k_0(X)$.

We obtain a homomorphism $\alpha_0 : K_0(C(X_{n_0})) \to K_1(B)$ by $\alpha_0([p_j']) = [z_j]$, $j = 1, 2, ..., k_0(X)$. Let $\alpha_0' = (\pi')_{*0} \circ \alpha_0$. Let $\alpha_{0,i} = (\pi_i)_{*0} \circ \alpha_0$ and $\alpha_{0,i}' = (\pi_i')_{*0} \circ \alpha_0$, $i = 1, 2, ..., R$. Note that

$$\alpha_0'([p_j']) = [z_j|_{Y_d'}], \ \ \alpha_{0,i}([p_j']) = [z_j|_{Y_i}] \text{ and } \alpha_{0,i}'([p_i']) = [z_j(\xi_i)] \tag{e 12.640}$$

$j = 1, 2, ..., k_0(X)$ and $i = 1, 2, ..., R$. Let $G_1 = [\psi_{n_0,\infty}](K_0(C(X_{n_0})))$.

Define $\Gamma : G_1 \to U(M_{m(X)+d+1}(B))/CU(M_{m(X)+d+1}(B))$ by $\Gamma([p_j]) = \bar{z}_j^*$, $j = 1, 2, ..., k_0(X)$. Define $\Gamma' : G_1 \to U(M_{m(X)+d+1}(B'))/CU(M_{m(X)+d+1}(B'))$ by $\Gamma'([p_j]) = \overline{z_j^*|_{Y_d'}}$, $j = 1, 2, ..., k_0(X)$. Define $\Gamma_i : G_1 \to U(M_{m(X)+d+1}(B_i))/CU(M_{m(X)+d+1}(B_i))$ by $\Gamma_i = \pi_i^\ddagger \circ \Gamma$ and $\Gamma_i' : G_1 \to U(M_{m(X)+d+1}(B_i'))/CU(M_{m(X)+d+1}(B_i'))$ by $\Gamma_i' = (\pi_i')^\ddagger \circ \Gamma$ and $i = 1, 2, ..., R$. Note that Γ, Γ', Γ_i and Γ_i' are compatible with $-\alpha_0$, $-\alpha_0'$, $-\alpha_{0,i}$ and $-\alpha_{0,i}'$, respectively. We note that

$$\Gamma'([p_j]) = \overline{z_j^*|_{Y_d'}}, \, \Gamma_i([p_j]) = \overline{z_j^*|_{Y_i}} \text{ and } \Gamma_i'([p_j]) = \overline{z_j^*(\xi_i)}, \tag{e 12.641}$$

$j = 1, 2, ..., k_0(X)$ and $i = 1, 2, ..., R$.

By the Universal Coefficient Theorem, there is $\alpha \in KK(C(X_{n_0}), B)$ such that $\alpha|_{K_i(C(X_{n_0}))} = \alpha_i$, $i = 0, 1$. It follows (using (e 12.635)) from 10.6 (for $\mathrm{dim}Y \le d$) that there is a unitary $U \in M_{K_4}(B)$ such that

$$\|[\varphi^{(K_4)}(g), \, U]\| < \delta_u/2 \text{ for all } g \in \mathcal{G}_u, \tag{e 12.642}$$

$$\mathrm{Bott}(\varphi^{(K_4)} \circ \psi_{n_0,\infty}, U) = -K_4\alpha \text{ and} \tag{e 12.643}$$

$$\mathrm{dist}(\mathrm{Bu}(\varphi, \, U)(x), \, K_4\Gamma(x)) < \gamma_0/64(d+1+m(X)) \text{ for all } x \in \mathcal{Q}. \tag{e 12.644}$$

Denote by $Z_{i,j} = (\pi_i(\bar{g}_{j,(1)}))^{(K_4)}$, $i = 1, 2, ..., R$ and $j = 1, 2, ..., k(X)$. In the following computation, we will identify $U(\xi_i)$, $W(\xi_i)$, $X_{1,j}(\xi_i)$ and $Z_{i,j}(\xi_i)$ with constant unitaries in $M_{m(X)}(B')$, when it makes sense. We also will use (2) and (4) above, as well as (e 12.620)-(e 12.622) in the following computation. We have

$$\mathrm{bott}_1((\pi_i \circ \varphi)^{(K_4)}, ((W(\xi_i)^{(K_4)}U(\xi_i))^*W|_{Y_i}^{(K_4)}U|_{Y_i}))(g_j) \tag{e 12.645}$$

$$= \mathrm{bott}_1(Z_{i,j}, U(\xi_i)^*(W(\xi_i)^*X_{1,j}|_{Y_j}X_{1,j}|_{Y_j}^*W|_{Y_i})^{(K_4)}U|_{Y_i}) \tag{e 12.646}$$

$$= \mathrm{bott}_1(Z_{i,j}, U(\xi_i)^*(W(\xi_i)^*X_{i,j}(\xi_i)X_{i,j}^*|_{Y_i}W|_{Y_i})^{(K_4)}U|_{Y_i}) \tag{e 12.647}$$

$$= \mathrm{bott}_1(Z_{i,j}, U(\xi_i)^*(W(\xi_i)^*X_{i,j}(\xi_i))^{(K_4)}) + \mathrm{bott}_1(Z_{i,j}, (X_{i,j}^*W)|_{Y_i}^{(K_4)}U|_{Y_i}) \tag{e 12.648}$$

$$= \mathrm{bott}_1(Z_{i,j}(\xi_i), U(\xi_i)^*(W(\xi_i)^*X_{i,j}(\xi_i))^{(K_4)}) \tag{e 12.649}$$

$$+\mathrm{bott}_1(Z_{i,j}, (X_{i,j}^*|_{Y_i}W|_{Y_i})^{(K_4)}) + \mathrm{bott}_1(Z_{i,j}, U|_{Y_i}) \tag{e 12.650}$$

$$= \mathrm{bott}_1(Z_{i,j}(\xi_i), U(\xi_i)^*) + \mathrm{bott}_1(Z_{i,j}(\xi_i), (W(\xi_i)^*X_{i,j}(\xi_i))^{(K_4)}) \tag{e 12.651}$$

$$+K_4\alpha_{1,i}(g_j') - K_4\alpha_{1,i}(g_j') \tag{e 12.652}$$

$$= K_4\alpha_{1,i}'(g_j') - K_4\alpha_{1,i}'(g_j') = 0. \tag{e 12.653}$$

Similarly, (put $Q_{0,j}=q_{0,j}^{(K_4)}$),

$$\begin{aligned}
&\mathrm{bott}_0((\pi_i\circ\varphi)^{(K_4)}, (W(\xi_i)^{(K_4)}U(\xi_i))^*W|_{Y_i}^{(K_4)}U|_{Y_i})([p_j]) && \text{(e 12.654)}\\
&= \mathrm{bott}_0(Q_{0,j}|_{Y_i}, U(\xi_i)^*(W(\xi_i)^*W|_{Y_i})^{(K_4)}U|_{Y_i}) && \text{(e 12.655)}\\
&= \mathrm{bott}_0(Q_{0,j}|_{Y_i}, U(\xi_i)^*(W(\xi_i)^*X_{0,j}|_{Y_i}X_{0,j}^*|_{Y_i}W|_{Y_i})^{(K_4)}U|_{Y_i}) && \text{(e 12.656)}\\
&= \mathrm{bott}_0(Q_{0,j}|_{Y_i}, U(\xi_i)^*(W(\xi_i)^*X_{0,j}(\xi_i)X_{0,j}^*|_{Y_i}W|_{Y_i'})^{(K_4)}U|_{Y_i}) && \text{(e 12.657)}\\
&= \mathrm{bott}_0(Q_{0,j}|_{Y_i}, U(\xi_i)^*(W(\xi_i)^*X_{0,j}(\xi_i))^{(K_4)}) && \text{(e 12.658)}\\
&\qquad +\mathrm{bott}_0(Q_{0,j}|_{Y_i}, (X_{0,j}^*|_{Y_i}W|_{Y_i})^{(K_4)}U|_{Y_i}) && \text{(e 12.659)}\\
&= \mathrm{bott}_0(Q_{0,j}|_{Y_i}, U(\xi_1)^*)+\mathrm{bott}_0(Q_{0,j}|_{Y_i}, (W(\xi_i)^*X_{0,j}(\xi_i))^{(K_4)}) && \text{(e 12.660)}\\
&\qquad +\mathrm{bott}_0(Q_{0,j}|_{Y_i}, (X_{0,j}^*|_{Y_i}W|_{Y_i})^{(K_4)})+\mathrm{bott}_0(Q_{0,j}|_{Y_i}, U|_{Y_i}) && \text{(e 12.661)}\\
&= \mathrm{bott}_0(Q_{0,j}(\xi_i), U(\xi_1)^*)+\mathrm{bott}_0(Q_{0,j}(\xi_i), (W(\xi_i)^*X_{0,j}(\xi_i))^{(K_4)}) && \text{(e 12.662)}\\
&\qquad +K_4\alpha_{0,j}([p_j'])-K_4\alpha_{0,j}([p_j']) && \text{(e 12.663)}\\
&= K_4\alpha_{0,j}'([p_j'])-K_4\alpha_{0,j}'([p_j'])=0. && \text{(e 12.664)}
\end{aligned}$$

Since $K_4x=0$ for all $x\in \mathrm{Tor}(K_i(C(X_{n_0}))$, $i=0,1$, we have

$$\mathrm{Bott}((\pi_i\circ\varphi\circ\psi_{n_0,\infty})^{(K_4)},\ (W(\xi_i)^{(K_4)}U(\xi_i))^*W|_{Y_i}^{(K_4)}U|_{Y_i})=0, \qquad \text{(e 12.665)}$$

$i=1,2,...,R$. It follows that

$$\mathrm{Bott}((\pi_i\circ\varphi)^{(K_4)},\ (W(\xi_i)^{(K_4)}U(\xi_i))^*W|_{Y_i}^{(K_4)}U|_{Y_i})|_{\mathcal{P}_0}=0, \qquad \text{(e 12.666)}$$

$i=1,2,...,R$.

We also estimate on Y_i, using (e 12.629), (e 12.642) and (4),

$$\begin{aligned}
&\|[\pi\circ\varphi^{(K_4)}(g),\ (W(\xi_i)^{(K_4)}U(\xi_i))^*W|_{Y_i}^{(K_4)}U|_{Y_i}]\| && \text{(e 12.667)}\\
&< \epsilon_1/4+\delta_u/2+\epsilon_1/4+\epsilon_1/4+\epsilon_1/4+\epsilon_1/4+\delta_u/2<\delta_1 && \text{(e 12.668)}
\end{aligned}$$

for all $g\in\mathcal{G}_u$.

For each i, there is $\Xi_j\in U(M_{K_4}(B))$ such that

$$\|\Xi_j-1\|<\delta_u/2 \ \text{ and } \ \Xi_jUQ_{0,j}U^*\Xi_j^*=Q_{0,j},\ \ j=1,2,...,k_0(X). \qquad \text{(e 12.669)}$$

Denote

$$P_{i,j}=(1-Q_{0,j})|_{Y_i} \ \text{ and } \ Q_{i,j}=Q_{0,j}|_{Y_i}.$$

$j=1,2,...,k_0(X)$ and $i=1,2,...,R$. By identifying $(\Xi_jU)(\xi_i)$, $(\Theta\bar{W})(\xi_i)$ and $X_{0,j}(\xi_i)$ with constant unitaries on Y_i, by (3) and (4) above, there is a unitary $\Xi_{i,j}\in M_{K_4}(B_i)$ such that

$$\|\Xi_{i,j}-1\|<\epsilon_1/4 \ \text{ and } \ \|[Q_{i,j},\ \Xi_{ij}(\Xi_jU)(\xi_i)^*((\Theta_i\bar{W}(\xi_i))^*\Theta_i|_{Y_i}\bar{W}|_{Y_i})^{(K_4)}(\Xi_jU)|_{Y_i}]\|=0,$$

$j=1,2,...,k_0(X)$ and $i=1,2,...,R$. Similarly, there is a unitary $\Xi_{i,j}'\in M_{K_4}(B_i)$ such that

$$\|\Xi_{i,j}'-1\|<\epsilon_1/4 \ \text{ and } \ \|[Q_{i,j},\ \Xi_{ij}(\Xi_jU)(\xi_i)^*((\Theta_i\bar{W}(\xi_i))^*X_{0,j}(\xi_i))^{(K_4)}\Xi_{i,j}']\|=0,$$

$j=1,2,...,k_0(X)$ and $i=1,2,...,R$. Set

$$P_{i,j}'=(1-Q_{0,j})(\xi_i) \ \text{ and } \ Q_{i,j}'=Q_{0,j}(\xi_i)$$

as constant projections. Define

$$\Omega_{i,j} = P_{i,j}+Q_{i,j}(X_{0,j}^*\Theta_i\bar{W})^{(K_4)}\Xi_jU)|_{Y_i} \text{ and} \tag{e 12.670}$$

$$\Omega_{i,j}' = P_{i,j}'+Q_{i,j}'(\Xi_jU)(\xi_i)^*((\Theta_i\bar{W}(\xi_i))^*X_{0,j}(\xi_i))^{(K_4)} \tag{e 12.671}$$

Then (see also (3) above)

$$\|\Omega_{i,j}'-(P_{i,j}+Q_{i,j}\Xi_{ij}(\Xi_jU)(\xi_i)^*((\Theta_i\bar{W}(\xi_i))^*X_{0,j}(\xi_i))^{(K_4)}\Xi_{i,j}'\|<\epsilon_1/2 \tag{e 12.672}$$

Thus, from above,

$$\|\Omega_{i,j}'\Omega_{i,j}-(P_{i,j}+Q_{i,j}\Xi_{ij}(\Xi_jU)(\xi_i)^*((\Theta_i\bar{W}(\xi_i))^*)^{(K_4)}(\Theta_i\bar{W}|_{Y_i})^{(K_4)}(\Xi_jU)|_{Y_i}\|<\epsilon_1 \tag{e 12.673}$$

We also have (see 2.15 and 2.21), by (e 12.644),

$$\mathrm{dist}(\mathrm{Bu}((\pi_i\circ\varphi)^{(K_4)},(\bar{W}(\xi_i)^{(K_4)}U(\xi_i))^*\bar{W}|_{Y_i}^{(K_4)}U|_{Y_i})([p_j])),\ \bar{1}) \tag{e 12.674}$$

$$<\mathrm{dist}(\overline{P_{i,j}+\bar{Q}_{i,j}(\Xi_iU)(\xi_i)^*((\Theta_i\bar{W}(\xi_i))^*\Theta_i|_{Y_i}\bar{W}|_{Y_i})^{(K_4)}(\Xi_iU)|_{Y_i}},\ \bar{1})+2\delta_u \tag{e 12.675}$$

$$<\mathrm{dist}(\overline{\Omega_{i,j}'\Omega_{i,j}},\ \bar{1})+\epsilon_1+2\delta_u \tag{e 12.676}$$

$$\le \mathrm{dist}(\overline{\Omega_{i,j}'},\bar{1})+\mathrm{dist}(\overline{\Omega_{i,j}},\bar{1})+\epsilon_1+2\delta_u \tag{e 12.677}$$

$$=\mathrm{dist}(\overline{(P_{i,j}+\bar{Q}_{i,j}(\Xi_iU)(\xi_i)^*(\Theta_i\bar{W}(\xi_i)X_{0,j}(\xi_i))^{(K_4)}},\ \bar{1}) \tag{e 12.678}$$

$$+\mathrm{dist}(\overline{(P_{i,j}+\bar{Q}_{i,j}(X_{0,j}^*\Theta_i|_{Y_i}\bar{W}|_{Y_i})^{(K_4)}(\Xi_iU)|_{Y_i}},\ \bar{1})+\epsilon_1+2\delta_u \tag{e 12.679}$$

$$=\mathrm{dist}(\overline{(P_{i,j}+\bar{Q}_{i,j}(\Theta_i\bar{W}(\xi_i)X_{0,j}(\xi_i))^{(K_4)}},\overline{(P_{i,j}+\bar{Q}_{i,j}(\Xi_iU)(\xi_i))}) \tag{e 12.680}$$

$$+\mathrm{dist}(\overline{(P_{i,j}+\bar{Q}_{i,j}(X_{0,j}^*\Theta_i|_{Y_i}\bar{W}|_{Y_i})^{(K_4)}},\overline{(P_{i,j}+\bar{Q}_{i,j}(\Xi_iU)^*|_{Y_i})}+\epsilon_1+2\delta_u \tag{e 12.681}$$

$$=\mathrm{dist}(\overline{(\overline{z_j^*}(\xi_i)^{(K_4)}},\overline{(P_{i,j}+\bar{Q}_{i,j}(\Xi_iU)(\xi_i))}) \tag{e 12.682}$$

$$+\mathrm{dist}(\overline{z_j}|_{Y_i}^{(K_4)},\overline{(P_{i,j}+\bar{Q}_{i,j}(\Xi_iU)^*|_{Y_i})}+\epsilon_1+2\delta_u \tag{e 12.683}$$

$$<\gamma_0/64(d+1+m(X))+\gamma_0/64(d+1+m(X)+\epsilon_1+2\delta_u \tag{e 12.684}$$

$$<\gamma_0/(d+1+m(X)),\ \ j=1,2,...,k_0(X) \text{ and } \ i=1,2,...,R. \tag{e 12.685}$$

Now we are ready to apply 11.1 (for $\dim Y\le d$) using (e 12.666), (e 12.674)-(e 12.685),(e 12.667) and (e 12.630). By 11.1, there is a continuous path of unitaries $\{V_i(t): t\in[0,1]\}\subset M_{K_1K_4}(B_i)$ such that

$$V_i(0)=(W(\xi_i)^{(K_1K_4)}U(\xi_i)^{(K_1)})^*W^{(K_1K_4)}|_{Y_i}U^{(K_1)}|_{Y_i},\ \ V(1)=1, \tag{e 12.686}$$

$$\text{and } \ \|[(\pi_i\circ\varphi)^{(K_1K_4)}(f),\,V_i(t)]\|<\epsilon/32 \text{ for all } t\in[0,1] \text{ and } f\in\mathcal{F}, \tag{e 12.687}$$

$i=1,2,...,R.$

Define $u\in M_{K_1K_4}(C)$ (in fact it should be in $M_{K_1K_4K_5}(C)$ but we replace $M_{K_5}(C)$ by C early on) as follows: $u(y)=W(y)^{(K_1K_4)}U^{(K_1)}(y)$ for $y\in Y_d'$. Note that D_i is homeomorphic to the $d+1$-dimensional open ball of radius 1. Each point of D_i is identified by a pair (x,t), where x is on $\partial D_i\cong S^{d+1}$, the boundary of D_j and t is distance from the point to the center ξ_i. Let $f_i:\partial D_i\to Y_i$ be the continuous map given by Y. Now define (note that $V_i(t)\in M_{K_1K_4}(B_i)$)

$$u_i(x,t)=W(\xi_i)^{(K_1K_4)}U(\xi_i)^{(K_1)}V_i(1-t)(f_i(x)) \tag{e 12.688}$$

Note that $u(x,1)=W^{(K_1K_4)}(f_i(x))U^{(K_1)}(f_i(x))$ for $x\in\partial D_i$ and $u_i(x,0)=W(\xi_i)^{(K_1K_4)}U(\xi_i)^{(K_1)}$. Define u on D_i as $u_i(x,t)$. Then $u\in M_{K_1K_4}(C)$, $u|_{Y_d}=(W^{(K_1K_4)}U^{(K_1)})|_{Y_d}$. Let $K=K_1K_4$. We have

$$\|u\varphi^{(K)}(f)u-\psi^{(K)}(f)\|_{Y_d} = \|W^{(K)}U^{(K_1)}\varphi^{(K)}(f)(U^*)^{(K_1)}(W^*)^{(K)}-\psi^{(K)}(f)\|_{Y_d} \tag{e 12.689}$$

$$< \delta_u/2+\|W^{(K)}\varphi^{(K)}(f)W^{(K)}-\psi^{(K)}(f)\||_{Y_d} \tag{e 12.690}$$

$$< \delta_u/2+\epsilon_1/4<\epsilon \text{ for all } f\in\mathcal{F}. \tag{e 12.691}$$

Moreover, for $y \in D_i$ and any $y' \in Y_i$, by applying (5) and (6) and by applying (e 12.688) and (e 12.687), we have

$$\begin{aligned}
&\|u(y)\varphi^{(K)}(f)(y)u^*(y) - \psi^{(K)}(f)(y)\| && \text{(e 12.692)}\\
&\quad < \|u(y)\varphi^{(K)}(f)(y')u^*(y) - \psi^{(K)}(f)(\xi_i)\| + 3\epsilon_1/4 && \text{(e 12.693)}\\
&\quad < \|W(\xi_i)^{(K)}U(\xi_i)^{(K_4)}\varphi^{(K)}(f)(y')(U(\xi_i)^*)^{(K_4)}(W(\xi_i)^*)^{(K)} - \psi^{(K)}(f)(\xi_i)\| && \text{(e 12.694)}\\
&\qquad +\epsilon/32 + 3\epsilon_1/4 && \text{(e 12.695)}\\
&\quad < \|W(\xi_i)^{(K)}\varphi^{(K)}(f)(y')(W(\xi_i)^*)^{(K)} - \psi^{(K)}(f)(\xi_i)\| + \delta_u/2 + \epsilon/32 + 3\epsilon_1/4 && \text{(e 12.696)}\\
&\quad < \|W(\xi_i)^{(K)}\varphi^{(K)}(f)(\xi_1)(W(\xi_i)^*)^{(K)} - \psi^{(K)}(f)(\xi_i)\| && \text{(e 12.697)}\\
&\qquad +\epsilon_1/4 + \delta_u/2 + \epsilon/32 + 3\epsilon_1/4 && \text{(e 12.698)}\\
&\quad < \|W(\xi_i)^{(K)}\varphi^{(K)}(f)(\xi_i)(W(\xi_i)^*)^{(K)} - \psi^{(K)}(f)(\xi_i)\| && \text{(e 12.699)}\\
&\qquad +\epsilon_1/4 + \delta_u/2 + \epsilon/32 + \epsilon_1 && \text{(e 12.700)}\\
&\quad < \epsilon_1/4 + \delta_u/2 + \epsilon/32 + 5\epsilon_1/4 < \epsilon \quad \text{for all } f \in \mathcal{F}. && \text{(e 12.701)}
\end{aligned}$$

It follows that

$$\|u\varphi^{(K)}(f)u^* - \psi^{(K)}(f)\| < \epsilon \ \text{for all } f \in \mathcal{F}. \qquad \text{(e 12.702)}$$

13 The reduction

Theorem 13.1. *The statement of* 10.4 *holds for all those compact subsets Y of a finite CW complex with dimension no more than d, where d is a non-negative integers.*

Proof. Let $\epsilon > 0$ and $\mathcal{F} \subset C(X)$ be a finite subset is given. Let $\Delta_1(r) = \Delta(r/3)/3$ for all $r \in (0,1)$. Let $\eta_1 > 0$ (in place of η), $\delta_1 > 0$ (in place of δ) $\gamma_1' > 0$ (in place of γ_1), $\gamma_2' > 0$ (in place of γ_2), $\mathcal{G} \subset C(X)$ be a finite subset, $\mathcal{P} \subset \underline{K}(C(X))$ be a finite subset, $\mathcal{H} \subset C(X)_{s.a.}$ be a finite subset, $\mathcal{V} \subset K_1(C(X)) \cap \mathcal{P}$, $N \geq 1$ be an integer and $K \geq 1$ be an integer required by 10.4 for ϵ, $\mathcal{F}$, Δ_1 and d.

Let $\eta = \eta_1/3$, $\delta = \delta_1/2$, $\gamma_1 = \gamma_1'/2$. Suppose that φ, ψ and $C = PM_m(C(Y))P$ satisfy the assumption for the above η, δ, γ_1, γ_2, $\mathcal{G}$, $\mathcal{P}$, $\mathcal{H}$, $\mathcal{V}$, N and K.

Suppose that $C = \lim_{n\to\infty}(C_n, \psi_n)$, where $C_n = P_nM_m(C(Y_n))P_n$, where Y_n is a finite CW complex of dimension no more than d. Let $\delta > \delta_0 > 0$ and $\mathcal{G}_0 \subset C$ be a finite subset. It follows from 6.7 that there exists an integer $n \geq 1$, a unital contractive completely positive linear map $r : C \to C_n$ and unital contractive completely positive linear maps $\Phi, \Psi : C(X) \to C_n$ such that $\Phi = r \circ \varphi$, $\Psi = r \circ \psi$,

$$\begin{aligned}
\|\psi_{n,\infty} \circ L(f) - \varphi(f)\| &< \delta_0 \ \text{for all } f \in \mathcal{G}, && \text{(e 13.703)}\\
\|\psi_{n,\infty} \circ r(g) - g\| &< \delta_0 \ \text{for all } f \in \mathcal{G}_0 \ \text{and} && \text{(e 13.704)}\\
\mu_{t\circ\Phi}(O_r),\ \mu_{t\circ\Psi}(O_r) &\geq \Delta(r/3)/3 \ \text{for all } t \in T(C_n) && \text{(e 13.705)}
\end{aligned}$$

for all $r \geq 17\eta_1/8$.

By choosing small δ_0 and large $\mathcal{G}_0$, we see that we reduce the general case to the case that Y is a finite CW complex and 10.4 applies.

□

Theorem 13.2. *Let A be a unital separable simple C^*-algebra which is tracially $\mathcal{I}_d$ for integer $d \geq 0$. Then $A \otimes Q$ has tracial rank at most one.*

Proof. Let $\epsilon > 0$, $a \in A \otimes Q_+ \setminus \{0\}$ and let $\mathcal{F} \subset A \otimes Q$ be a finite subset. We may assume that $1_A \in \mathcal{F}$. Note that A and $A \otimes Q$ has the strict comparison for positive elements. Let $b = \inf\{d_\tau(a) : \tau \in T(A)\}$. Then $b > 0$.

We write $Q = \lim_{n\to\infty}(M_{n!}, \imath_n)$, where $\imath_n : M_{n!} \to M_{(n+1)!}$ is a unital embedding defined by $\imath_n(x) = x \otimes 1_{M_{n+1}}$ for all $x \in M_{n!}$. To simplify notation, without loss of generality, we may assume that $\mathcal{F} \subset A \otimes M_{n!}$ for some integer $n \geq 1$. Denote $A_0 = A \otimes M_{n!}$. Since A_0 is tracially $\mathcal{I}_d$, there is a projection $e_0 \in A_0$ and a unital C^*-subalgebra $B_0 = EM_{r_0}(C(X))E$ with $1_{B_0} = e_0$, where X is a compact subset of a finite CW complex with dimension at most d, $r_0 \geq 1$ is an integer and $E \in M_{r_0}(C(X))$ such that

$$\|e_0x - xe_0\| < \epsilon/8 \text{ for all } x \in \mathcal{F}, \tag{e 13.706}$$
$$\mathrm{dist}(e_0xe_0, B_0) < \epsilon/8 \text{ for all } x \in \mathcal{F} \text{ and} \tag{e 13.707}$$
$$\tau(1 - e_0) < b/8 \text{ for all } \tau \in T(A_0). \tag{e 13.708}$$

We may assume that $E(x) \neq 0$ for all $x \in X$. Let $\mathcal{F}_1 \subset B_0$ be a finite subset such that

$$\mathrm{dist}(e_0xe_0, \mathcal{F}_1) < \epsilon/8 \text{ for all } x \in \mathcal{F}. \tag{e 13.709}$$

We may assume that $1_{B_0} \in \mathcal{F}_1$. We may assume that X is an infinite set (in fact, if B_0 can always be chosen to finite dimensional, then A is an AF algebra). To simplify notation, without loss of generality, we may assume that $\mathcal{F}$ and $\mathcal{F}_1$ are in the unit ball.

Put $A_1 = e_0A_0e_0$. Let $\imath' : B_0 \to A_1$ be the unital embedding. By 4.5, there is $r' \geq 1$, a projection $E' \in M_{r'}(B_0)$ and $W' \in M_{r'}(B_0)$ such that $E'M_{r'}(B_0)E' \cong M_{k_0}(C(X)) = C(X) \otimes M_{k_0}$ for some $k_0 \geq 1$ and $(W')^*1_{B_0}W' \leq E'$. Let $E = (W')E'(W')^*$. Then $EM_{r'}(B_0)E \cong C(X) \otimes M_{k_0}$ and $e_0 = 1_{B_0} \in EM_{r'}(B_0)E$. Let $A_2 = E((e_0Ae_0) \otimes M_{r'})E$. Let $e \in EM_r(B_0)E$ be a projection which may be identified with $1_{C(X)} \otimes e' \in C(X) \otimes M_{k_0}$, where $e' \in M_{k_0}$ is a minimum rank one projection. We also identify e with the projection in A_2. Put $B_1 = eM_{r'}(B_0)e$. Note that $B_1 \cong C(X)$. We will identify B_1 with $C(X)$ when it is convenient. Denote by $\imath : B_1 \to eA_2e$ be the embedding. Denote $A_3 = eA_2e$. We will identify $EM_{r'}(B_0)E$ with $M_{k_0}(B_1)$ and $M_{k_0}(eAe)$ with A_2. There exists a nondecreasing map $\Delta : (0,1) \to (0,1)$ such that

$$\mu_{\tau\circ\imath}(O_r) \geq \Delta(r) \text{ for all } \tau \in T(A_3) \tag{e 13.710}$$

and for all open balls O_r with radius $r > 0$. It follows from 4.1 that A_3 is also tracially $\mathcal{I}_d$. There exists a finite subset $\mathcal{F}_2$ in the unit ball of B_1 such that

$$\{(a_{i,j})_{k_0\times k_0} \in M_{k_0}(B_1) : a_{i,j} \in \mathcal{F}_2\} \supset \mathcal{F}_1 \tag{e 13.711}$$

(Here, again, we identify $EM_{r'}(B_0)E$ with $M_{k_0}(B_1)$).

Define $\Delta_1(r) = \Delta(r/3)/3$ for all $r \in (0,1)$. Let $\eta > 0$, $\gamma_1, \gamma_2 > 0$, $\mathcal{G} \subset B_1$ be a finite subset, $\mathcal{P} \subset \underline{K}(B_1)$ be a finite subset, $\mathcal{H} \subset (B_1)_{s.a.}$ be a finite subset, $\mathcal{V} \subset K_1(B_1) \cap \mathcal{P}$ be a finite subset, $N \geq$ and $K \geq 1$ be integers required by 13.1 for $\epsilon/2^8(r')^2$ (in place of ϵ), $\mathcal{F}_2$ (in place of $\mathcal{F}$) and Δ_1 (in place of Δ). Let $\mathcal{U} \in U(M_r(B_1))$ be a finite subset (for some integer $r \geq 1$) such that the image of $\mathcal{U}$ in $K_1(B_1)$ is $\mathcal{V}$.

Let $\{X_n\}$ be a decreasing sequence finite CW complexes such that $X = \cap_{n=1}^\infty X_n$ and let $s_n : C(X_n) \to C(X_{n+1})$ be the map defined by $s_n(f) = f|_{X_{n+1}}$ for $f \in C(X)$. Write $B_1 = \lim_{n\to\infty}(C(X_n), s_n)$. Choose an integer $L_1 \geq 1$ such that $1/L_1 < b/8$. Let $n_1 \geq 1$ (in place of n) be an integer, $q_1, q_2, ..., q_s \in C(X_{n_1})$ be mutually orthogonal projections $g_1, g_2, ..., g_k$, $G_0 \subset K_0(C(X))$, $N \geq 1$ be an integer and $\mathcal{G}_1 \subset U(M_l(B_1))/CU(M_l(B_1))$ (for some integer $l \geq 1$) a finite subset with $\overline{\mathcal{U}} \subset G_1$ be as required by 9.12 for $\epsilon/2^8(r')^2$ (in place of ϵ), $\gamma_1/4$ (in place of σ_1), $\gamma_2/4$ (in place σ_2), $\mathcal{G}$, $\mathcal{P}$, $\mathcal{H}$, $\mathcal{U}$, and L_1. We may assume that $r = l$ without loss of generality (by choosing the larger among them).

Choose $\delta_1 > 0$ and a finite subset $\mathcal{G}_1 \subset B_1$ such that, for any δ_1-$\mathcal{G}_1$-multiplicative contractive completely positive linear map L from B_1, $[L \circ s_{n_1,\infty}]$ is well defined on $\underline{K}(C(X_{n_1}))$. Let $q_1', q_2', ...q_k' \in K_0(C(X_{n_1}))$ such that $(s_{n_1,\infty})_{*0}(g_i') = g_i$, $i = 1, 2, ..., k$. We may also assume, by applying 10.3 of [34], that

$$|\rho_{C'}([L \circ s_{n_1,\infty}](g_i'))(\tau)| < 1/2N \ \text{ for all } \ \tau \in T(C'), \tag{e 13.712}$$

$i = 1, 2, ..., k$, for any δ_1-$\mathcal{G}_1$-multiplicative contractive completely positive linear map $L : B_1 \to C'$ for any unital C^*-algebra C' with $T(C') \neq \emptyset$. By choosing smaller δ_1 and larger $\mathcal{G}_1$, we may also assume that $[L]$ induces a well-defined homomorphism Λ' on G_1. Furthermore, we may assume that

$$[L \circ s_{n_1,\infty}](\xi) = \Pi(\Lambda'(g)) \tag{e 13.713}$$

for all $g \in G_1$ and $\xi \in K_1(C_{n_1})$ such that $g = (s_{n_1,\infty})_{*0}(\xi)$, provided L is a δ_1-$\mathcal{G}_1$-multiplicative contractive completely positive linear map.

Choose a set $\mathcal{F}_3 \subset B_1$ of $(2N+1)(d+1)$ mutually orthogonal positive elements. Since A_3 is simple and unital, there are $x_{f,1}, x_{f,2}, ..., x_{f,f(n)} \in A_3$ such that

$$\sum_{j=1}^{f(n)} x_{f,j}^* f x_{f,j} = e \ \text{ for all } \ f \in \mathcal{F}_4. \tag{e 13.714}$$

Let $N_0 = \{\max\{f(n) : f \in \mathcal{F}_3\} \max\{\|x_{f,j}\| : f \in \mathcal{F}_3, \ 1 \le j \le f(n)\}$. Let $\delta_2 = \min\{\delta/2, \delta_1/2, \epsilon/2^8(r')^2\}$ and let $\mathcal{G}_2 = \mathcal{F}_2 \cup \mathcal{G} \cup \mathcal{F}_3 \cup \mathcal{G}_1$.

Since A_3 is tracially $\mathcal{I}_d$, applying 6.9, there is a projection $e_1 \in A_3$, a unital C^*-subalgebra $C = PM_R(C(Y))P \in \mathcal{I}_d$ with $1_C = e_1$ such that

$$\|e_1 x - x e_1\| < \delta_1 \ \text{ for all } \ x \in \mathcal{G}_1, \tag{e 13.715}$$
$$\mathrm{dist}(e_1 x e_1, C) < \delta_1 \ \text{ for all } \ x \in \mathcal{G}_1, \tag{e 13.716}$$
$$\tau(1_{A_3} - e_1) < b/8 \ \text{ for all } \ \tau \in T(A_3), \tag{e 13.717}$$

and there exists a unital δ_1-$\mathcal{G}_1$-multiplicative contractive completely positive linear map $\Phi : B_1 \to C$ such that

$$\|\Phi(x) - e_1 x e_1\| < \delta_1 \ \text{ for all } \ x \in \mathcal{G}_1 \ \text{ and} \tag{e 13.718}$$
$$\mu_{\tau \circ \Phi}(O_r) \ge \Delta_1(r) \ \text{ for all } \ \tau \in T(e_1 A_3 e_1) \tag{e 13.719}$$

for all open balls O_r with radius $r \ge \eta$. We may also assume that there is a projection $E^{(0)} \in M_{k_0}(e_1 A_3 e_1)$ such that $E^{(0)} \le e_0$ and

$$\|e_0(e_1 \otimes 1_{M_{k_0}})e_0 - E^{(0)}\| < \epsilon/2^8 \tag{e 13.720}$$

(Note that $e_0 = 1_{B_0} \in EM_{r'}(B_1)E = M_{k_0}(C(X)))$. Note also we identify $(e_1 \otimes 1_{M_{k_0}})$ with a projection $E' \le E$ in A_2.

We also have, for $x \in \mathcal{F}_1$, by (e 13.711) and above,

$$\|E^{(0)}x - xE^{(0)}\| < \epsilon/2^7 + \|e_0(e_1 \otimes 1_{M_{k_0}})e_0 x - x e_0(e_1 \otimes 1_{M_{k_0}})e_0\| \tag{e 13.721}$$
$$= \epsilon/2^7 + \|e_0(e_1 \otimes 1_{M_{k_0}})x - x(e_1 \otimes 1_{M_{k_0}})e_0\| \tag{e 13.722}$$
$$< \epsilon/2^7 + (k_0)^2\delta_1 < 3\epsilon/2^8 \tag{e 13.723}$$

Similarly,

$$\|E^{(0)}xE^{(0)} - (\Phi\otimes \mathrm{id}_{M_{k_0}})(x)\| \;\;<\;\; \epsilon/2^6 \;\text{ for all }\; x\in \mathcal{F}_1. \tag{e 13.724}$$

Since $\mathcal{G}_1\supset \mathcal{F}_3$, we conclude that, for each $y\in Y$, $\mathrm{rank}P(y)>2N(d+1)$. Let $\kappa=[\Phi\circ s_{n_1,\infty}]$. It follows from the above construction, by 9.12, that there is a unital $\epsilon/2^8(r')^2$-$\mathcal{G}$-multiplicative contractive completely positive linear map $\Psi: B_1\to C$, mutually orthogonal projections $Q_0, Q_1, ..., Q_{L_1}, Q_{L_1+1}$ such that $Q_0, Q_1, ..., Q_{L_1}$ are mutually equivalent, $P=\sum_{i=1}^{L_1} Q_i$,

$$[\Psi\circ s_{n_1,\infty}] \;\;=\;\; [\Phi\circ s_{n_1,\infty}], \tag{e 13.725}$$
$$\mathrm{dist}(\Psi^{\ddagger}(x), L^{\ddagger}(x)) \;\;<\;\; \gamma_1/2 \;\text{ for all }\; x\in \overline{\mathcal{U}} \;\text{ and} \tag{e 13.726}$$
$$|\tau\circ\Psi(a)-\tau\circ\Phi(a)| \;\;<\;\; \gamma_2 \;\text{ for all }\; a\in\mathcal{H} \;\text{ and }\;\; \text{for all }\; \tau\in T(C), \tag{e 13.727}$$

and $\Psi=\Psi_0\oplus\overbrace{\Psi_1\oplus\Psi_1\oplus\cdots\oplus\Psi_1}^{L_1}\oplus\Psi_2$, where $\Psi_0: B_1\to Q_0CQ_0$, $\Psi_1=\psi_1\circ\varphi_0$, $\psi_1: C(J)\to Q_1CQ_1$ is a unital homomorphism, $\varphi_0: B_1\to C(J)$ is a unital $\epsilon/2^8(r')^2$-$\mathcal{G}$-multiplicative contractive completely positive linear map $\Phi_2=\psi_2\circ\varphi_0$, $\psi_2: C(J)\to Q_{L_1+1}CQ_{L_1+1}$ is a unital homomorphism, and where J is a finite disjoint union of intervals.

It follows from 13.1 that there is an integer $K\ge 1$ and a unitary $U\in A_3\otimes M_K$ such that

$$\|U^*\Phi^{(K)}(f)U-\Psi^{(K)}(f)\|<\epsilon/2^8(r')^2 \;\text{ for all }\; f\in\mathcal{F}_2. \tag{e 13.728}$$

Choose $n_1'\ge 1$ such that $K|n_1'$. Note that $e_1A_3e_1\subset e_1A_3e_1\otimes M_K\otimes M_{(n_1')!/(n_1)!K}$. With that, we may write $\Phi^{(K)}(f)=\Phi(f)\otimes 1_{M_K}$ and $\Psi^{(K)}(f)=\Psi(f)\otimes 1_{M_K}$. It follows that (working in $A_2\otimes M_K$)

$$\|(\Phi\otimes\mathrm{id}_{M_{k_0}})(x)\otimes 1_{M_K}-\bar{U}(\Psi\otimes\mathrm{id}_{M_{k_0}})(x)\otimes 1_{M_K})\bar{U}^*\|<\epsilon/2^8 \;\text{ for all }\; x\in\mathcal{F}_1, \tag{e 13.729}$$

where $\bar{U}=U\otimes 1_{M_{k_0}}$. Put

$$C_1=(\overbrace{\psi_1\oplus\psi_1\oplus\cdots\oplus\psi_1}^{L_1}\oplus\psi_2)(C(J)) \;\text{ and }\; \Psi'=\overbrace{\Psi_1\oplus\Psi_1\oplus\cdots\oplus\Psi_1}^{L_1}\oplus\Psi_2. \tag{e 13.730}$$

There is a projection $E^{(1)}\in M_{k_0}(C_1)$ such that

$$\|(\Psi'\otimes\mathrm{id}_{M_{k_0}})(1_{B_0})-E^{(1)}\|<\epsilon/2^7. \tag{e 13.731}$$

Put $E_2=\bar{U}(E^{(1)}\otimes 1_{M_K})\bar{U}^*$. Thus, by (e 13.729) and (e 13.720),

$$\|(E^{(0)}\otimes 1_{M_K})E_2-E_2\|<3\epsilon/2^8. \tag{e 13.732}$$

There is a projection $E_3\le (E^{(0)}\otimes 1_{M_K})$ and

$$\|E_3-E_2\|<3\epsilon/2^7. \tag{e 13.733}$$

It follows that there is a unitary $U_1\in M_{k_0}(e_1A_3e_1\otimes M_K)=E'A_2E'\otimes M_K$ with $\|U_1-1\|<3\epsilon/2^7$ such that $U_1^*E_2U_1=E_3$. Put $W=\bar{U}^*U_1$. Let

$$C_2=W^*(E^{(1)}\otimes 1_{M_K})(M_{k_0}(C_1)\otimes M_K)(E^{(1)}\otimes 1_{M_K})W.$$

Since $E_3 \in M_{k_0}(C_1) \otimes M_K$, $C_2 \in \mathcal{I}^{(1)}$. We estimate that, for $x \in \mathcal{F}_1$, by identifying x with $x \otimes 1_{M_K}$,

$$\begin{aligned}
\|E_3x - xE_3\| &< \|E_2x - xE_2\| + 3\epsilon/2^6 && \text{(e 13.734)}\\
&= \|E_2(E' \otimes 1_{M_K})x - x(E' \otimes 1_K)E_2\| + 3\epsilon/2^6 && \text{(e 13.735)}\\
&< \|E_2(E'xE' \otimes 1_{M_K}) - (E'xE' \otimes 1_{M_K})E_2\| + \epsilon/2^7 + 3\epsilon/2^6 && \text{(e 13.736)}\\
&< \|E_2(\Phi \otimes \mathrm{id}_{M_K})(x) - (\Phi \otimes \mathrm{id}_{M_K})(x)E_2\| + 2(k_0)^2\delta_1 + 5\epsilon/2^6 && \text{(e 13.737)}\\
&< \|E_2(\bar{U}((\Psi \otimes \mathrm{id}_{M_k})(x) - (\Psi \otimes \mathrm{id}_{M_k})(x))\bar{U}^*E_2\| + 2\epsilon/2^8 + 6\epsilon/2^6 && \text{(e 13.738)}\\
&< 8\epsilon/2^6 = \epsilon/2^3. && \text{(e 13.739)}
\end{aligned}$$

Similarly, for $x \in \mathcal{F}_1$,

$$\begin{aligned}
&\|E_3xE_3 - (\Psi' \otimes \mathrm{id}_{k_0})(x) \otimes 1_{M_K}\| && \text{(e 13.740)}\\
&< 3\epsilon/2^6 + \|E_2xE_2 - E_2e_1xe_1E_2\| + && \text{(e 13.741)}\\
&\quad \|E_2e_1xe_1E_2 - W^*((\Psi' \otimes \mathrm{id}_{k_0})(x) \otimes 1_{M_K})W\| && \text{(e 13.742)}\\
&< 3\epsilon/2^6 + (k_0)^2\delta_1 && \text{(e 13.743)}\\
&\quad + \|E_2(\Phi \otimes \mathrm{id}_{M_{k_0}})(x) \otimes 1_{M_K})E_2 - W^*((\Psi' \otimes \mathrm{id}_{k_0})(x) \otimes 1_{M_K})W\| && \text{(e 13.744)}\\
&< 7\epsilon/2^6 + 6\epsilon/2^7 && \text{(e 13.745)}\\
&\quad + \|E_2(\Phi \otimes \mathrm{id}_{M_{k_0}})(x) \otimes 1_{M_K})E_2 - \bar{U}((\Psi' \otimes \mathrm{id}_{k_0})(x) \otimes 1_{M_K})\bar{U}^*\| && \text{(e 13.746)}\\
&< 19\epsilon/2^6 + \epsilon/2^6 && \text{(e 13.747)}\\
&\quad + \|E_2[(\Phi \otimes \mathrm{id}_{M_{k_0}})(x) \otimes 1_{M_K} - \bar{U}((\Psi \otimes \mathrm{id}_{k_0})(x) \otimes 1_{M_K})\bar{U}^*)E_2\| && \text{(e 13.748)}\\
&< 20\epsilon/2^6 + \epsilon/2^8 = 24\epsilon/2^6 = 3\epsilon/2^3. && \text{(e 13.749)}
\end{aligned}$$

Therefore

$$\mathrm{dist}(E_3xE_3,\, C_2) < 3\epsilon/2^3 \text{ for all } x \in \mathcal{F}_1. \qquad \text{(e 13.750)}$$

Note that $E_3 \le E^{(0)} \otimes 1_{M_K}$. It follows that $E_3 \otimes 1_{M_K} \le e_0 \otimes 1_{M_K}$. Note that we identify x with $x \otimes 1_{M_K}$. So It follows that, for all $y \in \mathcal{F}$, $E_3(y \otimes 1_{M_K})E_3 = E_3(e_0 \otimes 1_{M_K})(y \otimes 1_{M_K})(e_0 \otimes 1_{M_K})E_3$,

$$\mathrm{dist}(E_3(y \otimes 1_{M_K})E_3, C_2) < 3\epsilon/2^3 + 2\epsilon/8 < \epsilon. \qquad \text{(e 13.751)}$$

Since $\mathcal{F} \subset A_0 \subset A_0 \otimes M_K \otimes M_{n_1!/n!K}$, $C_2 \subset A_0 \otimes M_K \otimes M_{n_1!/n!K} \subset A$. One may write (e 13.751) as

$$\mathrm{dist}(E_3yE_3, C_2) < \epsilon \text{ for all } x \in \mathcal{F} \qquad \text{(e 13.752)}$$

and by (e 13.734)-(e 13.739), one may write that

$$\|E_3y - yE_3\| < \epsilon/8 + \epsilon/2^3 + \epsilon/8 < \epsilon \text{ for all } y \in \mathcal{F}. \qquad \text{(e 13.753)}$$

By identify e_0 with $e_0 \otimes 1_{M_K}$ in A, we also have, by the choice of L_1, that

$$\tau((1 - e_0) + (e_0 - E_3)) < b/8 + b/8 = b/4 \text{ for all } \tau \in T(A). \qquad \text{(e 13.754)}$$

Thus

$$\max\{\tau(1 - E_3) : \tau \in T(A)\} < b = \inf\{d_\tau(a) : \tau \in T(A)\}. \qquad \text{(e 13.755)}$$

It follows from 5.5 that

$$1 - E_3 \lesssim a. \qquad \text{(e 13.756)}$$

Therefore, from (e 13.753), (e 13.752) and (e 13.756), A has tracial rank at most one.

□

Theorem 13.3. *Let A be a unital separable simple C^*-algebra which is tracially $\mathcal{I}_d$ for some integer $d \geq 0$. Suppose that A satisfies the UCT. Then A has tracial rank at most one and is isomorphic to a unital simple AH-algebra with no dimension growth.*

Proof. It follows 13.2 that A is rationally tracial rank at most one. It follows from 5.6 that $K_0(A)$ is weakly unperforated Riezs group. Moreover, by 6.4, the map from $T(A)$ to $S_1(K_0(A))$ maps the $\partial_e(T(A))$ onto $\partial_e(K_0(A))$. Thus, by [56], there is a unital simple AH-algebra C with no dimension growth has the same Elliott invariant as that of A. Since A is assumed to satisfy the UCT, by the classification theorem in [35], $A \otimes \mathcal{Z} \cong C$. But, by 8.4, A is $\mathcal{Z}$-stable. Therefore $A \cong C$. This proves, in particular, A has tracial rank at most one. □

Proof of Theorem 1.2

Proof. This is immediately consequence of 13.3 □

The proof of Theorem 1.1

Proof. This is an immediate corollary of 13.3. There is $d \geq 0$ such that A is tracially $\mathcal{I}_d$. It follows from 13.2 that A has tracial rank at most one. Note that, since A is locally AH, A also satisfies the UCT.

□

14 Appendix

In the definition of 3.5 and 3.6, we use $\mathcal{I}^{(k)}$ and $\mathcal{I}_k$ as model classes of C^*-algebras of rank k. In general, however, one could have more general C^*-algebras as defined below.

Definition 14.1. Denote by $\overline{\mathcal{I}_k}$ the class of C^*-algebras with the form $PM_r(C(X))P$, where X is a compact metric space with covering dimension k, $r \geq 1$ and $P \in M_r(C(X))$ is a projection.

However the following proposition shows that, by replacing $\mathcal{I}_k$ by $\overline{\mathcal{I}}_k$, one will not make any gain.

Proposition 14.2. *Let $C = PM_r(C(X))P \in \overline{\mathcal{I}_k}$. Then, for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset C$, there exists a C^*-subalgebra $C_1 \subset C$ such that $C_1 \in \mathcal{I}_k$, with $C_1 = QM_r(C(Y))Q$, where Y is a compact subset of a finite CW complex of dimension at most k, $Q \in M_r(C(Y))$ is a projection such that*

$$\mathrm{dist}(a, C_1) < \epsilon \ \textit{for all}\ a \in \mathcal{F} \ \text{and} \tag{e 14.757}$$

$$\inf\{\mathrm{rank}\, Q(y) : y \in Y\} = \inf\{\mathrm{rank}\, P(x) : x \in X\} \tag{e 14.758}$$

Proof. There is a sequence of finite CW complexes $\{X_n\}$ with covering dimension k such that $C(X) = \lim_{n\to\infty}(C(X_n), \varphi_n)$, where $\varphi_n : C(X_n) \to C(X_{n+1})$ is a unital homomorphism. Let $\varphi_{n,\infty} : C(X_n) \to C(X)$ be the unital homomorphism induced by the inductive limit system. There is a compact subset $Y_n \subset X_n$ such that $\varphi_{n,\infty}(C(X_n)) = C(Y_n)$. Note that $C(Y_n) \subset C(Y_{n+1}) \subset C(X)$. Moreover, $C(X) = \overline{\cup_{n=1}^{\infty} C(Y_n)}$. Let $\imath_n : C(Y_n) \to C(X)$ be the imbedding. Denote by $s_n : X \to Y_n$ the surjective continuous map such that $\imath_n(f)(x) = f(s_n(x))$ for all $f \in C(Y_n)$ (and $x \in X$). Denote again by $\imath_n$ the extension from $M_r(C(Y_n))$ to $M_r(C(X))$.

Now let $1 > \epsilon > 0$ and let $\mathcal{F} \subset PM_r(C(X))P \subset M_r(C(X))$ be a finite subset. There is an integer $n \geq 1$, a projection $Q \in M_r(C(Y_n))$ and a finite subset $\mathcal{G} \subset QM_r(C(Y_n))Q$ such that

$$\|P - Q\| < \epsilon/2 \ \text{and}\ \mathrm{dist}(x, \mathcal{G}) < \epsilon/2$$

for all $x \in \mathcal{F}$. It follows that $\imath_n(Q)$ has the same rank as that of P at every point of $x \in X$. But for each $y \in Y_n$, there exists $x \in X$ such that $y = s_n(x)$. Therefore (e 14.758) also holds.

$\square$

Lemma 14.3. *Let G be a group, let $a, b \in G$ and let $k \geq 2$ be an integer. Then there are $2(k-1)$ commutators $c_1, c_2, ..., c_{2(k-1)} \in G$ such that*

$$(ab)^k = a^k b^k (\prod_{j=1}^{2(k-1)} c_{2(k-1)+(j-1)}).$$

Proof. Note that

$$abab = ab^2 a(a^{-1}b^{-1}ab) \text{ for all } a, b \in G. \tag{e 14.759}$$

Let $c_1 = (a^{-1}b^{-1}ab)$. Therefore

$$ab^2 a c_1 = a(ab^2)((b^2)^{-1}a^{-1}b^2 a)c_1. \tag{e 14.760}$$

Let $c_2 = ((b^2)^{-1}a^{-1}b^2 a)$. Thus

$$abab = a^2 b^2 (c_2 c_1).$$

This proves the lemma for $k = 2$. Suppose that the lemma holds for $1, 2, ..., k-1$. Then

$$(ab)^k = ab(a^{k-1}b^{k-1})(c_{2(k-2)}c_{2(k-2)-1}\cdots c_1), \tag{e 14.761}$$

where $c_1, c_2, ..., c_{2(k-2)}$ are commutators. As (e 14.759),

$$ab(a^{k-1}b^{k-1}) = ab(b^{k-1})(a^{k-1})(a^{-(k-1)}b^{-(k-1)}a^{k-1}b^{k-1}) \tag{e 14.762}$$

Let $c_{2(k-1)-1} = (a^{-(k-1)}b^{-(k-1)}a^{k-1}b^{k-1})$. Further,

$$ab^k a^{k-1} = aa^{k-1}b^k(b^{-k}a^{-(k-1)}b^k a^{k-1}). \tag{e 14.763}$$

Let $c_{2(k-1)} = (b^{-k}a^{-(k-1)}b^k a^{k-1})$. Then

$$(ab)^k = a^k b^k (c_{2(k-1)}c_{2(k-1)-1}\cdots c_1. \tag{e 14.764}$$

This completes the induction. $\square$

Lemma 14.4. (1) *Let A be a unital C^*-algebra and let $u \in U(A)$. If $1/2 > \epsilon > 0$ and $\|u^k - v\| < \epsilon$ for some integer $k \geq 1$ and $v \in U_0(A) \cup CU(A)$, then there exists $v_1 \in CU(A)$ and $u_1 \in U(A)$ such that*

$$\|u - u_1\| < \epsilon/k \text{ and } u_1^k \in CU(A).$$

Moreover, there are $2(k-1)$ commutators $c_1, c_2, ..., c_{2(k-1)}$ such that

$$u_1^k = v(\prod_{j=1}^{2(k-1)} c_j).$$

(2) *If A is a unital infinite dimensional simple C^*-algebra with (SP), then there is $u \in CU(A)$ such that $sp(u) = \mathbb{T}$.*

Proof. For (1), there is $h \in A_{s.a.}$ such that

$$\exp(ih) = u^k v^* \text{ and } \|h\| < 2\arcsin(\epsilon/2). \tag{e 14.765}$$

Let $u_1 = u\exp(-ih/k)$. Then

$$u_1^k \in CU(A). \tag{e 14.766}$$

One also has that

$$\|u - u_1\| < \epsilon/k.$$

By 14.3,

$$u_1^k = vc_{2(k-1)}c_{2(k-1)-1}\cdots c_1$$

for some commutators $c_1, c_2, ..., c_{2(k-1)}$ of $U(A)$.

For (2), let e_1, e_2 be two non-zero mutually orthogonal and mutually equivalent projections. Since e_1Ae_1 is an infinite dimensional simple C^*-subalgebra, one obtains a unitary $u_1 \in e_1Ae_1$ such that $sp(u_1) = \mathbb{T}$. Let $w \in A$ such that $w^*w = e_1$ and $ww^* = e_2$. Put $z = (1-e_1-e_2)+w+w^*$. Then $z \in U(A)$. Define $u = (1-e_1-e_2)+wu_1w^*+u_1$. One verifies that $sp(z) = \mathbb{T}$ and $z \in CU(A)$. □

The following is known and has been used a number of times. We include here for convenience and completeness.

Proposition 14.5. *Let A be a separable unital C^*-algebra and let $\mathcal{U} \subset U(A)$ be a finite subset. Then, for any $\epsilon > 0$, there exists $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: for any unital C^*-algebra B and any δ-$\mathcal{G}$-multiplicative contractive completely positive linear map $L : A \to B$, there exists a homomorphism $\lambda : G_{\mathcal{U}} \to U(B)/CU(B)$ such that*

$$\mathrm{dist}(\overline{\langle L_n(u)\rangle}, \lambda(\bar{u})) < \epsilon$$

for all $u \in \mathcal{U}$, where $G_{\mathcal{U}}$ is the subgroup generated by $\{\bar{u} : u \in \mathcal{U}\}$.

Proof. Suppose that the proposition is false. Then, there is $\epsilon_0 > 0$, a finite subset $\mathcal{U} \subset U(A)$, and sequence of decreasing numbers $\delta_n > 0$ with $\lim_{n\to\infty} \delta_n = 0$ and a sequence of an increasing finite subsets $\mathcal{G}_n \subset A$ with $\cup_{n=1}^{\infty}\mathcal{G}_n$ being dense in A, a sequence of unital C^*-algebras B_n and a sequence of δ_n-$\mathcal{G}_n$-multiplicative contractive completely positive linear maps $L_n : A \to B_n$ such that

$$\inf\{\sup_{u\in\mathcal{U}} \mathrm{dist}(\overline{\langle L_n(u)\rangle}, \lambda(\bar{u}) \geq \epsilon_0, \tag{e 14.767}$$

where the infimum is taken among all homomorphisms $\lambda : G_{\mathcal{U}} \to U(B_n)/CU(B_n)$. Define $\Psi : A \to \prod_{n=1}^{\infty} B_n$ by $\Psi(x) = \{L_n(x)\}$ for all $x \in A$. Let $Q = \prod_{n=1}^{\infty} B_n/\oplus_{n=1}^{\infty} B_n$ and let $\Pi : \prod_{n=1}^{\infty} B_n \to Q$ be the quotient map. Then $\Pi \circ \Psi : A \to Q$ is a homomorphism. Therefore it induces a homomorphism $(\Pi\circ\Psi)^{\ddagger} : U(A)/CU(A) \to U(Q)/CU(Q)$. Fix an integer $k \geq 1$. Suppose $z \in U(Q)/CU(Q)$ such that $z^k = 0$. There exists a unitary $u_z \in U(Q)$ such that $\bar{u}_z = z$. There are $w_1, w_2, ..., w_N \in U(Q)$ which are commutators such that $u_z^k = \prod_{j=1}^{N} w_j$. Suppose $w_j = a_jb_ja_j^*b_j^*$, $a_j, b_j \in U(Q)$. There are unitaries $x_n, a_{j(n)}, b_{j(n)} \in B_n$ such that

$$\Pi(\{x_n\}) = u_z, \ \Pi(\{a_{j(n)}\}) = a_j \text{ and } \Pi(\{b_{j(n)}\}) = b_j, \tag{e 14.768}$$

$j = 1, 2, ..., N$. It follows that

$$\lim_{n\to\infty} \|x_n^k - \prod_{j=1}^{N} a_{j(n)}b_{j(n)}a_{j(n)}^*b_{j(n)}^*\| = 0. \tag{e 14.769}$$

It follows from 14.4 that there exists a sequence of unitaries $y_n \in B_n$ such that

$$y_n^k = \prod_{j=1}^N a_{j(n)} b_{j(n)} a_{j(n)}^* b_{j(n)}^* (\prod_{i=1}^{2(k-1)} v_{i(n)}),$$

where $v_{j(n)}$ are unitaries in B_n which are commutators, and

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{e 14.770}$$

In particular, $\Pi(\{y_n\}) = u_z$. Note that

$$\{\prod_{j=1}^N a_{j(n)} b_{j(n)} a_{j(n)}^* b_{j(n)}^* (\prod_{i=1}^k v_{i(n)})\} \in CU(\prod_{n=1}^\infty B_n).$$

We have just shown that every finite subgroup of $U(Q)/CU(Q)$ lifts to a finite subgroup of the same order. This implies that there exists a homomorphism $\gamma : G_{\mathcal U} \to U(\prod_{n=1}^\infty B_n)/CU(\oplus_{n=1}^\infty B_n)$ such that $\Pi^\ddagger \circ \gamma = (\Pi \circ \Psi)^\ddagger|_{G_{\mathcal U}}$. Let $\pi_n : \prod_{n=1}^\infty B_n \to B_n$ be the projection on the n-coordinate. Define $\lambda_n : G_{\mathcal U} \to U(B_n)/CU(B_n)$ by $\lambda_n = \pi_n^\ddagger \circ \gamma$, $n = 1, 2, ...$ There exists $n_0 \ge 1$, for all $n \ge n_0$,

$$\mathrm{dist}(\overline{\langle L_n(u)\rangle}, \pi_n^\ddagger \circ \gamma(\bar u)) < \epsilon_0/2 \tag{e 14.771}$$

for all $u \in \mathcal U$. This is a contradiction.

□

References

[1] B. Blackadar, *K-theory for operator algebras*, Mathematical Sciences Research Institute Publications, 5. Springer-Verlag, New York, 1986. viii+338 pp. ISBN: 0-387-96391-X.

[2] B. Blackadar and D. Handelman, *Dimension functions and traces on C^*-algebras*, J. Funct. Anal. **45** (1982), 297-340.

[3] B. Blackadar and E. Kirchberg, *Generalized inductive limits of finite-dimensional C^*-algebras*, Math. Ann. **307** (1997), 343-380.

[4] O. Bratteli, G. A. Elliott, D. Evans and A. Kishimoto, *Homotopy of a pair of approximately commuting unitaries in a simple C^*-algebra*, J. Funct. Anal. **160** (1998), 466C523.

[5] N. Brown, *Invariant means and finite representation theory of C^*-algebras*, Mem. Amer. Math. Soc. **184** (2006), no. 865, viii+105 pp.

[6] N. Brown, F. Perera and A. Toms, *The Cuntz semigroup, the Elliott conjecture, and dimension functions on C^*-algebras*, J. Reine Angew. Math. **621** (2008), 191-211.

[7] N. Brown and A. Toms, *Three applications of the Cuntz semigroup*, Int. Math. Res. Not. IMRN 2007, no. 19, Art. ID rnm068, 14 pp

[8] J. Cuntz and G. K. Pedersen, *Equivalence and traces on C^*-algebras*, J. Funct. Anal. **33** (1979), 135-164.

[9] M. Dadarlat, *On the topology of the Kasparov groups and its applications*, J. Funct. Anal. **228** (2005), 394-418.

[10] M. Dadarlat and S. Eilers, *Approximate homogeneity is not a local property*, J. Reine Angew. Math. **507** (1999), 1-13.

[11] M. Dadarlat and S. Eilers, *On the classification of nuclear C^*-algebras*, Proc. London Math. Soc. **85** (2002), 168-210.

[12] M. Dadarlat and T. Loring, *K-homology, asymptotic representations, and unsuspended E-theory*, J. Funct. Anal. **126** (1994), 367-383.

[13] M. Dadarlat, G. Nagy, A. Nmethi and C. Pasnicu, *Reduction of topological stable rank in inductive limits of C^*-algebras*, Pacific J. Math. **153** (1992), 267-276.

[14] G. A. Elliott, *Dimension groups with torsion*, Internat. J. Math. **1** (1990), 361-380.

[15] G. A. Elliott and G. Gong, *On the classification of C^*-algebras of real rank zero. II*, Ann. of Math. **144** (1996), 497610,

[16] G. A. Elliott, G. Gong and L. Li, *On the classification of simple inductive limit C^*-algebras. II, The isomorphism theorem*, Invent. Math. **168** (2007), 249-320

[17] R. Exel and T. A. Loring, *Extending cellular cohomology to C^*-algebras*, Trans. Amer. Math. Soc. **329** (1992), 141160.

[18] G. Gong, *On the classification of simple inductive limit C^*-algebras. I, The reduction theorem*, Doc. Math. **7** (2002), 255-461.

[19] L. Li, *Simple inductive limit C^*-algebras: spectra and approximations by interval algebras*, J. Reine Angew. Math. **507** (1999), 5779.

[20] H. Lin, *Approximation by normal elements with finite spectra in C^*-algebras of real rank zero*, Pacific J. Math. 173 (1996), no. 2, 443-489,

[21] H. Lin, *Classification of simple tracially AF C^*-algebras*, Canad. J. Math. **53** (2001), 161-194.

[22] H. Lin, *Embedding an AH-algebra into a simple C^*-algebra with prescribed KK-data*, K-Theory **24** (2001), 135-156.

[23] H. Lin, *An introduction to the classification of amenable C^*-algebras*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001. xii+320 pp. ISBN: 981-02-4680-3.

[24] H. Lin, *The tracial topological rank of C^*-algebras*, Proc. London Math. Soc. **83** (2001), 199-234.

[25] H. Lin, *Stable approximate unitary equivalence of homomorphisms*, J. Operator Theory **47** (2002), 343-378.

[26] H. Lin, *Traces and simple C^*-algebras with tracial topological rank zero*, J. Reine Angew. Math. **568** (2004), 99-137.

[27] H. Lin, *Classification of simple C^*-algebras of tracial topological rank zero*, Duke Math. J. **125** (2004), 91-119.

[28] H. Lin, *An approximate universal coefficient theorem*, Trans. Amer. Math. Soc. **357** (2005), 375-405

[29] H. Lin, *Simple nuclear C^*-algebras of tracial topological rank one*, J. Funct. Anal. **251** (2007), 601-679.

[30] H. Lin, *Asymptotically unitary equivalence and asymptotically inner automorphisms*, Amer. J. Math. **131** (2009), 1589-1677,

[31] H. Lin, *Approximate homotopy of homomorphisms from $C(X)$ into a simple C^*-algebra*, Mem. Amer. Math. Soc. **205** (2010), no. 963, vi+131 pp

[32] H. Lin, *Localizing the Elliott Conjecture at Strongly Self-absorbing C*-algebras –An Appendix*, arXiv:0709.1654.

[33] H. Lin, *Homotopy of unitaries in simple C^*-algebras with tracial rank one*, J. Funct. Anal. **258** (2010), 822-882.

[34] H. Lin, *Approximate unitary equivalence in simple C^*-algebras of tracial rank one*, Trans. Amer. Math. Soc., to appear (arXiv:0801.2929).

[35] H. Lin, *Asymptotically unitary equivalence and classification of simple amenable C*-algebras*, Inven. Math. **183**, (2011), 385-450.

[36] H. Lin, *Homomorphisms from AH-algebras*, preprint, arXiv:1102.4631.

[37] H. Lin, and Z. Niu, *Lifting KK-elements, asymptotic unitary equivalence and classification of simple C^*-algebras*, Adv. Math. **219** (2008), 1729-1769.

[38] H. Lin and N. Phillips, *Almost multiplicative morphisms and the Cuntz algebra $\mathcal{O}_2$*, Internat. J. Math. **6** (1995), 625-643.

[39] H. Matui and Y. Sato, *Strict comparison and Z-absorption of nuclear C*-algebras*, preprint (arXiv:1111.1637).

[40] P. Ng and W. Winter, *Nuclear dimension and the corona factorization property* Int. Math. Res. Not. IMRN 2010, no. 2, 261-278

[41] F. Perera and A. Toms, *Recasting the Elliott conjecture*, Math. Ann. **338** (2007), 669-702.

[42] N. C. Phillips, *Reduction of exponential rank in direct limits of C^*-algebras*, Canad. J. Math. **46** (1994), 818-853.

[43] N. C. Phillips, *A survey of exponential rank*, C^*-algebras: 19431993 (San Antonio, TX, 1993), 352-399, Contemp. Math., 167, Amer. Math. Soc., Providence, RI, 1994.

[44] N. C. Phillips, *The C^* projective length of n-homogeneous C^*-algebras*, J. Operator Theory **31** (1994), 253-276.

[45] N. C. Phillips, *How many exponentials?*, Amer. J. Math. **116** (1994), 513-543.

[46] N. C. Phillips, *Factorization problems in the invertible group of a homogeneous C^*-algebra*, Pacific J. Math. **174** (1996), 215-246.

[47] M. Rieffel, *The homotopy groups of the unitary groups of noncommutative tori*, J. Operator Theory **17** (1987), 237-254.

[48] J. Ringrose, *Exponential length and exponential rank in C^*-algebras*, Proc. Roy. Soc. Edinburgh Sect. A **121** (1992), 55-61

[49] M. Rørdam, *On the structure of simple C^*-algebras tensored with a UHF-algebra*, J. Funct. Anal. **100** (1991), 1-17.

[50] M. Rørdam, *On the structure of simple C^*-algebras tensored with a UHF-algebra. II*, J. Funct. Anal. **107** (1992), 255-269

[51] L. Robert and L. Santiago, *Classification of C^*-homomorphisms from $C_0(0,1]$ to a C^*-algebra*, J. Funct. Anal. **258** (2010), 869-892

[52] K. Thomsen, *Traces, unitary characters and crossed products by* **Z**, Publ. Res. Inst. Math. Sci. **31** (1995), 1011-1029.

[53] A. Toms, *Stability in the Cuntz semigroup of a commutative C^*-algebra*, Proc. Lond. Math. Soc. (3) **96** (2008), 1-25.

[54] A. Toms, *K-theoretic rigidity and slow dimension growth*, preprint, arXiv:0910.2061.

[55] A. Toms and W. Winter, *Minimal dynamics and the classification of C^*-algebras*, Proc. Natl. Acad. Sci. USA 106 (2009), no. 40, 16942-16943,

[56] J. Villadsen, *The range of the Elliott invariant of the simple AH-algebras with slow dimension growth*, K-Theory **15** (1998), 1-12

[57] D. Voiculescu, *A note on quasidiagonal operators*, Topics in operator theory, 265-274, Oper. Theory Adv. Appl., 32, Birkhuser, Basel, 1988.

[58] W. Winter, *On the classification of simple $\mathcal{Z}$-stable C^*-algebras with real rank zero and finite decomposition rank*, J. London Math. Soc. (2) **74** (2006), 167-183.

[59] W. Winter, *Localizing the Elliott conjecture at strongly self-absorbing C*-algebras*, preprint, arXiv:0708.0283.

[60] W. Winter, *Nuclear dimension and Z-stability of perfect C*-algebras*, preprint, arXiv:1006.2731.